Math 165: Honors Calculus I Name:____________________________

Final Exam *Dec. 15, 1995*

There are 15 problems worth a total of 165 points.

1. (15 pts) Give complete definitions for the following concepts.

   a) $\sum_{i=1}^{n} a_i$

   b) A function f is integrable on $[a, b]$.

   c) $\lim_{x \to p} f(x) = A$.

   d) A function f is differentiable at p.

   e) A function f is convex on $[a, b]$.

2. (15 pts) State the following theorems precisely.

   a) LINEARITY THEOREM FOR INTEGRALS.

   b) MEAN VALUE THEOREM FOR INTEGRALS.

   c) INTERMEDIATE VALUE THEOREM.

   d) CHAIN RULE.

   e) MEAN VALUE THEOREM FOR DERIVATIVES.

3. (15 pts) Calculate the following.

a) $\int_0^2 [x^2]\, dx$ where $[u]$ is the greatest integer $\leq u$.

b) $\int_1^x t^3 + (t-1)^{1/3}\, dt,\ x \geq 1.$

c) The average value of the function $f(x) = x(4 - x^2)$ on the interval $[0, 2]$.

d) $\int_{-4}^{4} \frac{x^3 + x}{x^8 + x^2 + 1}\, dx$

4. (15 pts) Compute the following limits or prove they do not exist.

a) $\lim_{x \to \pi/4} \dfrac{\sin(x) - \cos(x)}{x - \pi/4}$ (Hint: $x = (x - \pi/4) + \pi/4$).

b) $\lim_{x \to 0} \dfrac{1 - \sqrt{1 - x}}{x}$.

c) $\lim_{x \to 0} \dfrac{(1 + x)^n - (1 - x)^n}{x}$ where n is a fixed positive integer.

d) $\lim_{x \to 1} (x^2 - x) f(x)$ where $f(x) = \begin{cases} x & x > 1 \\ -x & x < 1 \end{cases}$.

5. (15 pts) Compute the following derivatives.

a) $\dfrac{d}{dx}\left[x^3(1-x^2)^{50}\right]$.

b) $\dfrac{d}{dx}\sqrt{1+x^2}$

c) $\dfrac{d}{dx}\cos(\sin(x^3))$

d) $\dfrac{dy}{dx}$ where y is defined implicitly by the equation $y^4 + x^2y^3 + 1 = 0$.

6. (10 pts) Consider the sequence $\frac{5}{1}, \frac{19}{5}, \frac{65}{19}, \frac{211}{65}, \ldots,$ defined recursively by

$$a_1 = 5, \quad a_{n+1} = 5 - \frac{6}{a_n}, \ n \geq 1$$

Use induction to prove that

$$a_n = \frac{3^{n+1} - 2^{n+1}}{3^n - 2^n}, \ n \geq 1$$

7. (10 pts) Find the largest interval I containing $x = 1$ on which the function $f(x) = \dfrac{1}{1+x^2}$ has an inverse. Give a formula for the corresponding inverse function, $f^{-1}(x)$, as a function of x. Sketch the graph of f on I and the graph of f^{-1} on its domain.

8. (5 pts) Find an expression for the area between the graphs of the functions $f(x) = x^2 - 2x$ and $g(x) = 1 - x$ on the interval $[0, 3]$. Write the answer as a sum of integrals without absolute values—do not evaluate the integrals.

9. (10 pts) Use the *definition of a limit* to prove that $\lim_{x \to 1} x^2 = 1$.

10. (10 pts) Prove that if a function f is differentiable at p, then f is continuous at p.

11. (10 pts) Let

$$f(x) = \begin{cases} 6\sqrt{x} - 5, & \text{for } x < 1 \\ x^3, & \text{for } x \geq 1 \end{cases}$$

Prove that $f'(1) = 3$. You must use the definition of a derivative and should examine one-sided limits.

12. (5 pts) Find the equation of the line tangent to the curve defined by $y = x^4 - x^3 + x^2 - x + 1$ at the point $(1, 1)$.

13. (10 pts) If a and b are legs of a right triangle whose hypotenuse is 1, find the largest value of $2a + b$.

14. (10 pts) Let $f(x) = 3x^4 + 4x^3 - 12x^2$.

a) Determine the intervals on which f is increasing and decreasing.

b) Determine the relative extrema of f.

c) Determine the intervals on which f is convex and concave.

d) Sketch the graph of f.

15. (10 pts)

a) State ROLLE'S THEOREM.

b) Use BOLZANO'S THEOREM and ROLLE'S THEOREM to prove that the function $f(x) = x^3 - x^2 + 1$ has *exactly one* root in the interval $[-1, 0]$.