September 9, 1997

I wish to make a remark concerning the shifting of indices in a sum. A sum:

$$\sum_{k=0}^{n-1} c_k$$

can also be written as:

$$\sum_{k=1}^{n} c_{k-1}.$$

This simple trick is sometimes useful as we shall see in the proof of the binomial theorem.

Solutions (# 4 p.44) Use induction to prove the binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k b^{n-k}.$$

Proof. If $n = 1, LHS = a + b$,

$$RHS = \left(\begin{array}{c} 1 \\ 0 \end{array} \right) a^0 b^1 + \left(\begin{array}{c} 1 \\ 1 \end{array} \right) a^1 b^0 = a + b = LHS.$$

Assume that the Theorem is valid for n we want to show that it is also valid for $n+1$:

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \left(\begin{array}{c} n+1 \\ k \end{array} \right) a^k b^{n+1-k}.$$

We have:

$$\begin{aligned} LHS &= (a+b)(a+b)^n \\ &= (a+b) \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k b^{n-k} \\ &= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{k+1} b^{n-k} + \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k b^{n+1-k}. \end{aligned}$$

The first sum on the right above can be written as:

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{k+1} b^{n-k} = a^{n+1} + \sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) a^{k+1} b^{n-k}.$$

The second sum on the right above can be written as:

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k} = \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + b^{n+1}.$$

Thus

$$LHS = a^{n+1} + \sum_{k=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + b^{n+1}.$$

We use the trick of shifting indices to write the first sum above as:

$$\begin{aligned}
\sum_{k=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k} &= \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n-(k-1)} \\
&= \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n+1-k}.
\end{aligned}$$

Thus

$$\begin{aligned}
(a+b)^n &= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \{ \binom{n}{k-1} + \binom{n}{k} \} a^k b^{n+1-k} + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + b^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}.
\end{aligned}$$

We have used the identity:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

which is verified by a direct calculation:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(n+1)!}{k!(n-k+1)!} \\
&= \left(\begin{array}{c} n+1 \\ k \end{array} \right).
\end{aligned}
$$

If we set $a = b = 1$ in the binomial theorem we obtain the identity:

$$2^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right).$$

On the other hand, if we take $a = -1$ and $b = 1$ then we get

$$\sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) = 0.$$

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