More examples on induction

Example 2

$$1+3+5+...+(2n-1)=n^2$$

Proof. For $n=1, LHS=1, RHS=1^2=1$. Assume that identity holds for $n=k$:

$$1+3+5+...+(2k-1)=k^2.$$

Then for $n=k+1$,

$$LHS = 1+3+5+...+(2k-1)+(2(k+1)-1)=k^2+2k+1=(k+1)^2=RHS.$$

Example 3

$$1=1$$

$$1-4=-(1+2)$$

$$1-4+9=1+2+3$$

$$1-4+9-16=-(1+2+3+4)$$

Guess the general formula and prove it.

$$Guess: \quad 1-4+9-16+...+(-1)^{n-1}n^2=(-1)^{n-1}\frac{n(n+1)}{2}$$

Proof. For $n=1$, obviously true. Assume that the identity holds for $n=k$:

$$1-4+9-16+...+(-1)^{k-1}k^2=(-1)^{k-1}\frac{k(k+1)}{2}.$$

Then

$$\begin{aligned} LHS &= 1-4+9-16+...+(-1)^{k-1}k^2+(-1)^k(k+1)^2 \\ &= (-1)^{k-1}\frac{k(k+1)}{2}+(-1)^k(k+1)^2 \\ &= (-1)^k\frac{2(k+1)^2-k(k+1)}{2} \\ &= (-1)^k\frac{(k+1)(k+2)}{2}. \end{aligned}$$

HW: p.35 # 1c, 3

Example 4 General Properties of Sums:

$$\sum_{k=1}^{n}(a_n + b_n) = \sum_{k=1}^{n} a_n + \sum_{k=1}^{n} b_n \quad \textit{(additive property)}$$

$$\sum_{k=1}^{n} ca_n = c\sum_{k=1}^{n} a_n \quad \textit{(homogeneous property)}$$

$$\sum_{k=1}^{n} ca_n = c\sum_{k=1}^{n} a_n \quad \textit{(telescoping property)}$$

These properties can be used to prove:

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n}\{k^2 - (k-1)^2\} = n^2$$

by the additive and telescoping property.

$$\sum_{k=1}^{n} k^2 = ?$$

To figure out what the answer is observe that

$$k^2 = \frac{k^3 - (k-1)^3 + 3k - 1}{3}.$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} \frac{k^3 - (k-1)^3 + 3k - 1}{3} \\
&= \frac{1}{3}\sum_{k=1}^{n}(k^3 - (k-1)^3) + \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \\
&= \frac{n^3}{3} + \frac{n(n+1)}{2} - \frac{n}{3} \\
&= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.
\end{aligned}$$

The main point is this. If we already guess the correct formula then usually the validity of the formula can be checked by induction; however the much more difficult part is to come up with the correct formula.

HW: p.40 #7, 8

Solutions (# 7) Find the sum: $\sum_{i=1}^{n} i^3$. Since

$$(i+1)^4 - i^4 = 4i^3 + 6i^2 + 4i + 1$$

hence

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\{\sum_{i=1}^{n}((i+1)^4 - i^4) - 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i - n.$$

By the telescoping property:

$$\frac{1}{4}\sum_{i=1}^{n}((i+1)^4 - i^4) = \frac{1}{4}\{(n+1)^4 - 1\},$$

and by the previous examples:

$$\frac{1}{4}\{6\sum_{i=1}^{n} i^2\} = \frac{3}{2}\{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\}$$

and

$$\frac{1}{4}\{4\sum_{i=1}^{n} i\} = \{\frac{n^2}{2} + \frac{n}{2}\}.$$

Thus

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}\{(n+1)^4 - 1\} - \frac{3}{2}\{\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\} - \{\frac{n^2}{2} + \frac{n}{2}\} - \frac{n}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n}{6}.$$

(# 8) (a) $\sum_{k=0}^{n} = (1 - x^{n+1})/(1-x),\ x \neq 1.$

Observe that, by the telescoping property:

$$(1-x)\sum_{k=0}^{n} x^k = \sum_{k=0}^{n}(x^k - x^{k+1}) = 1 - x^{n+1}.$$

The formula now follows by dividing both sides by $1-x$.

(b) If $x = 1$ then

$$\sum_{k=0}^{n} x^k = 1^0 + 1^1 + ... + 1^n = n + 1.$$

Lemma If $a \geq 0$ then $|x| \leq a$ if and only if $-a \leq x \leq a$.

Theorem(Triangle Inequality) For any real numbers $a_1, ..., a_n$ we have:

$$|\sum_{i=1}^{n} \leq a_i| \leq \sum_{i=1}^{n} |a_i|^2.$$

Proof. Theorem is clearly true if $n = 1$. If $n = 2$, we have, by Lemma:

$$-|a_1| \leq a_1 \leq |a_1|, \ -|a_2| \leq a_2 \leq |a_2|.$$

Adding the two inequalities we obtain:

$$-(|a_1| + |a_2|) \leq a_1 + a_2 \leq |a_1| + |a_2|.$$

By Lemma this last inequality is equivalent to

$$|a_1 + a_2| \leq |a_1| + |a_2|.$$

The case of general n follows by induction. QED

Theorem (Cauchy-Schwarz Inequality) If $a_1, ..., a_n$ and $b_1, ..., b_n$ are arbitrary real numbers. Then

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2).$$

In fact, we have:

$$(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) - (\sum_{i=1}^{n} a_i b_i)^2 = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2.$$

Proof. Theorem is clearly true if $n = 1$. For $n = 2$,

$$(a_1 b_1 + a_2 b_2)^2 = (a_1 b_1)^2 + (a_2 b_2)^2 + 2 a_1 b_1 a_2 b_2$$

while

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1 b_1)^2 + (a_2 b_2)^2 + a_1^2 b_2^2 + a_2^2 b_1^2.$$

Thus

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2 = a_1^2 b_2^2 + a_2^2 b_1^2 - 2 a_1 b_1 a_2 b_2 = (a_1 b_2 - a_2 b_1)^2$$

is non-negative and is zero if and only if $a_1b_2 = a_2b_1$.

Assuming that the Theorem holds for all $n \leq k$ and consider the case $n = k+1$:

$$(\sum_{i=1}^{k+1} a_ib_i)^2 \leq (\sum_{i=1}^{k+1} a_i^2)(\sum_{i=1}^{k+1} b_i^2)?$$

Write the LHS as

$$((\sum_{i=1}^{k} a_ib_i) + a_{k+1}b_{k+1})^2 = (\sum_{i=1}^{k} a_ib_i)^2 + (a_{k+1}b_{k+1})^2 + 2a_{k+1}b_{k+1}\sum_{i=1}^{k} a_ib_i$$

and write the RHS as

$$\begin{aligned}
&((\sum_{i=1}^{k} a_i^2) + a_{k+1}^2)((\sum_{i=1}^{k} b_i^2) + b_{k+1}^2) \\
&= (\sum_{i=1}^{k} a_i^2)(\sum_{i=1}^{k} b_i^2) + a_{k+1}^2(\sum_{i=1}^{k} b_i^2) + b_{k+1}^2(\sum_{i=1}^{k} a_i^2) + a_{k+1}^2b_{k+1}^2.
\end{aligned}$$

Then the difference

$$\begin{aligned}
&RHS - LHS \\
&= (\sum_{i=1}^{k} a_i^2)(\sum_{i=1}^{k} b_i^2) - (\sum_{i=1}^{k} a_ib_i)^2 + a_{k+1}^2(\sum_{i=1}^{k} b_i^2) + b_{k+1}^2(\sum_{i=1}^{k} a_i^2) - 2a_{k+1}b_{k+1}\sum_{i=1}^{k} a_ib_i \\
&= \sum_{1\leq i<j\leq k} (a_ib_j - a_jb_i)^2 + a_{k+1}^2(\sum_{i=1}^{k} b_i^2) + b_{k+1}^2(\sum_{i=1}^{k} a_i^2) - 2a_{k+1}b_{k+1}\sum_{i=1}^{k} a_ib_i \\
&= \sum_{1\leq i<j\leq k} (a_ib_j - a_jb_i)^2 + \sum_{i=1}^{k}(a_ib_{k+1} - a_{k+1}b_i)^2 \\
&= \sum_{1\leq i<j\leq k+1} (a_ib_j - a_jb_i)^2.
\end{aligned}$$

We have used the induction hypothesis (i.e. the case $n = k$ is valid) and the elementary identity:

$$\sum_{i=1}^{k}(a_ib_{k+1} - a_{k+1}b_i)^2 = b_{k+1}^2\sum_{i=1}^{k} a_i^2 - 2a_{k+1}b_{k+1}\sum_{i=1}^{k} a_ib_i + a_{k+1}^2\sum_{i=1}^{k} b_i^2$$

QED

HW p. 43 # 1 (h), 1 (i), 1 (j); p.44 # 2, 3, 4