September 14, 1997

Example p. 45, # 13

(a) Let p be a positive integer. Prove that:

$$b^p - a^p = (b-a)(b^{p-1} + b^{p-2}a + ... + ba^{p-2} + a^{p-1}).$$

Proof. Expand the RHS by direct calculation:

$$\begin{aligned} RHS = b^p &+ b^{p-1}a + b^{p-2}a^2 + ... + b^2a^{p-2} + ba^{p-1} \\ &- b^{p-1}a - b^{p-2}a^2 - ... - b^2a^{p-2} - ba^{p-1} - a^p = b^p - a^p. \end{aligned}$$

(b) Let p and n be positive integers. Use (a) to show that:

$$n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p.$$

Proof. Set $b = n+1$ and $a = n$ in part (a). Then

$$(n+1)^p - n^p = ((n+1)^{p-1} + (n+1)^{p-2}n + ... + (n+1)n^{p-2} + n^{p-1}).$$

The RHS above consists of $p+1$ terms and each term satisfies the condition:

$$n^p \leq (n+1)^i n^{p-i} \leq (n+1)^p.$$

Hence

$$(p+1)n^p \leq (n+1)^p - n^p \leq (p+1)(n+1)^p$$

and (b) is obtained by dividing through by $p+1$.

(c) Use induction to prove that:

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p.$$

Proof. If $n = 1$ then LHS $= 0$ the middle term is $1/(p+1)$ and the RHS $= 1$. Thus the inequality is verified for $n = 1$. Assume now that the inequality is valid for n we want to show that it is also valid for $n+1$. For $n+1$ the LHS is, by assumption

$$LHS = \sum_{k=1}^{n+1-1} k^p = \sum_{k=1}^{n-1} k^p + n^p < \frac{n^{p+1}}{p+1} + n^p.$$

By (b), we conclude that

$$LHS < \frac{n^{p+1}}{p+1} + \frac{(n+1)^{p+1} - n^{p+1}}{p+1} = \frac{(n+1)^{p+1}}{p+1}.$$

On the other hand the RHS (for $n+1$) is by assumption:

$$RHS = \sum_{k=1}^{n+1} k^p = \sum_{k=1}^{n} k^p + (n+1)^p > \frac{n^{p+1}}{p+1} + (n+1)^p.$$

Hence, by (b):

$$RHS > \frac{n^{p+1}}{p+1} + (n+1)^p > \frac{n^{p+1}}{p+1} + \frac{(n+1)^{p+1} - n^{p+1}}{p+1} = \frac{(n+1)^{p+1}}{p+1}.$$

We have shown in class (Theorem 1.12, p. 77) that a monotone function $f(x)$ defined on a closed interval is integrable. Thus the function $f(x) = x^p, p$ any positive integer is integrable on the interval $[0, b], b > 0$. We now show, with the help of the preceding example that (Theorem 1.23, p. 79-80):

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1}.$$

Proof. As in the proof of Theorem 1.12, we consider the partition of $[0, b]$:

$$x_0 = 0,\ x_1 = \frac{b}{n},\ ...,\ x_k = \frac{kb}{n},\ ...,\ x_n = b.$$

Thus $f(x) = x^p$ is approximated from below by the step function

$$s_n = x_{k-1}^p = (\frac{kb}{n})^p \text{ on } (x_k, x_{k+1})$$

$k = 0, ..., n-1$ and from above by the step function:

$$t_n = x_{k+1}^p = (\frac{(k+1)b}{n})^p \text{ on } (x_k, x_{k+1})$$

$k = 0, ..., n-1$. By the definition of the integral of a step function we have:

$$\int_0^b s_n dx = \frac{b}{n} \sum_{k=0}^{n-1} (\frac{kb}{n})^p = \frac{b}{n} \sum_{k=1}^{n-1} (\frac{kb}{n})^p = \frac{b^{p+1}}{n^{p+1}} \sum_{k=1}^{n-1} k^p$$

(because $kb/n = 0$ if $k = 0$) and

$$\int_0^b t_n dx = \frac{b}{n}\sum_{k=0}^{n-1}(\frac{(k+1)b}{n})^p = \frac{b}{n}\sum_{k=1}^{n}(\frac{kb}{n})^p = \frac{b^{p+1}}{n^{p+1}}\sum_{k=1}^{n}k^p$$

by the trick of shifting indices (cf. the notes on September 9). By the preceding example

$$\int_0^b s_n dx < \frac{b^{p+1}}{n^{p+1}}\frac{n^{p+1}}{p+1} = \frac{b^{p+1}}{p+1} < \int_0^b t_n dx. \qquad (1)$$

On the other hand,

$$0 \le \int_0^b t_n dx - \int_0^b s_n dx = \frac{b^{p+1}}{n^{p+1}}\sum_{k=1}^{n}k^p - \frac{b^{p+1}}{n^{p+1}}\sum_{k=1}^{n-1}k^p = \frac{b^{p+1}}{n^{p+1}}n^p = \frac{b^{p+1}}{n}$$

which tends to 0 as $n \to \infty$, i.e.

$$\lim_{n\to\infty}\int_0^b t_n dx = \lim_{n\to\infty}\int_0^b s_n dx.$$

This together with the estimate (1) imply that

$$\int_0^b x^p dx = \lim_{n\to\infty}\int_0^b t_n dx = \lim_{n\to\infty}\int_0^b s_n dx = \frac{b^{p+1}}{p+1}.$$

HW p. 70 # 6, 8, 9; p. 83 # 21, 22, 23, 24