MATH 165: HONORS CALCULUS I
ASSIGNMENT 9 SOLUTIONS

Problem 1 p.45, #13

a) $b^p - a^p = (b-a)\sum_{k=0}^{p-1} b^{p-k-1}a^k$.

Proof.

$$\begin{aligned}(b-a)\sum_{k=0}^{p-1} b^{p-k-1}a^k &= \sum_{k=0}^{p-1} bb^{p-k-1}a^k - ab^{p-k-1}a^k \\ &= \sum_{k=0}^{p-1} b^{p-k}a^k - b^{p-(k+1)}a^{k+1} \\ &= b^{p-0}a^0 - b^{p-(p-1+1)}a^{p-1+1} \quad \text{[telescoping sum]} \\ &= b^p - a^p\end{aligned}$$

□

b) $n^p < \dfrac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p$

Proof. Use the formula from a) for the case $p+1$ and plug in $b = n+1$ and $a = n$. This gives

$$(n+1)^{p+1} - n^{p+1} = (n+1-n)\sum_{k=0}^{p}(n+1)^{p-k}n^k = \sum_{k=0}^{p}(n+1)^{p-k}n^k$$

Apply the inequality $n^k < (n+1)^k$ for $k = 1, \ldots, p$ on the right hand side to get

$$(n+1)^{p+1} - n^{p+1} < \sum_{k=0}^{p}(n+1)^{p-k}(n+1)^k = \sum_{k=0}^{p}(n+1)^p$$

The right hand side consists of $(p+1)$ copies of $(n+1)^p$ so we get

$$(n+1)^{p+1} - n^{p+1} < (p+1)(n+1)^p$$

The other inequality is similar. Apply the inequality $(n+1)^k > n^k$ for $k = 1, \dots, p$ on the right hand side to get

$$\begin{aligned}(n+1)^{p+1} - n^{p+1} &= \sum_{k=0}^{p} (n+1)^{p-k} n^k \\ &> \sum_{k=0}^{p} n^{p-k} n^k = \sum_{k=0}^{p} n^p \\ &= (p+1)n^p\end{aligned}$$

□

c) $\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p$ (*)

Proof. Use induction on n. The formula is true for $n = 1$:

$$0 = \sum_{k=1}^{0} k^p < \frac{1^{p+1}}{p+1} < \sum_{k=1}^{1} k^p$$

We assume the inequalities (*) are true for n, and prove they hold for $n+1$. The left hand side of (*) for $n+1$ looks like:

$$\begin{aligned}\sum_{k=1}^{n} k^p &= \left(\sum_{k=1}^{n-1} k^p\right) + n^p \\ &< \frac{n^{p+1}}{p+1} + n^p \quad \text{[induction hypothesis]} \\ &< \frac{(n+1)^{p+1}}{p+1} \quad \text{[by part b)]}\end{aligned}$$

so the left inequality is true for $n+1$. The right hand side of (*) for $n+1$ is:

$$\begin{aligned}\sum_{k=1}^{n+1} k^p &= \left(\sum_{k=1}^{n} k^p\right) + (n+1)^p \\ &> \frac{n^{p+1}}{p+1} + (n+1)^p \quad \text{[induction hypothesis]} \\ &> \frac{(n+1)^{p+1}}{p+1} \quad \text{[by part b)]}\end{aligned}$$

□

Problem 2 Let A and B be non-empty sets of real numbers bounded above and below, respectively.
a) If $k < \sup A$ then there is a number $a \in A$ such that $k < a$.

Proof. Suppose statement a) is false. Then there is some $k < \sup A$ such that $k \geq a$ for all $a \in A$. Therefore, k is an upper bound for A. Since $\sup A$ is the *least* upper bound for A, it must be less than k: $\sup A \leq k$. But this contradicts $k < \sup A$. Therefore statement a) must be true. □

b) If $h > \inf B$ then there is a number $b \in B$ such that $b < h$.

Proof. Suppose statement b) is false. Then there is some $h > \inf B$ such that $b \geq h$ for all $b \in B$. Therefore, h is a lower bound for B. Since $\inf B$ is the *greatest* lower bound for B, it must be greater than h: $\inf B \geq h$. But this contradicts $h > \inf B$. Therefore, statement b) must be true. □

Problem 3 Let A and B be non-empty sets of real numbers with A contained in B.
a) If A and B are bounded above, then $\sup A \leq \sup B$.

Proof. Let $k = \sup B$ and suppose $k < \sup A$. By 2a), there is a number $a \in A$ such that $k < a$. Since $k = \sup B \geq b$ for all $b \in B$, the number a cannot be in B. But this contradicts the assumption that B contains A. Since assuming $k < \sup A$ leads to a contradiction, it must be that $\sup A \leq k$, i.e., $\sup A \leq \sup B$. □

b) If A and B are bounded below, then $\inf B \leq \inf A$.

Proof. Let $h = \inf B$ and suppose $h > \inf A$. By 2b), there is a number $a \in A$ such that $a < h$. Since $h = \inf B \leq b$ for all b in B, the number a cannot be in B. But this contradicts the assumption that B contains A. Since assuming $h > \inf A$ leads to a contradiction, it must be that $h \leq \inf A$, i.e., $\inf B \leq \inf A$. □

Problem 4 For $0 \leq x \leq 1$ define

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

a) If s and t are step functions satisfying $s(x) \leq f(x) \leq t(x)$ for all $0 \leq x \leq 1$, then $s(x) \leq 0$ and $t(x) \geq 1$, except possibly at partition points.

Proof. Let $P = \{x_0, x_1, ..., x_n\}$ be a common refinement of partitions for s and t so that s and t are constant on the open subintervals of P: $s(x) = s_k$ and $t(x) = t_k$ for $x_{k-1} < x < x_k$, $k = 1, \ldots, n$. By assumption, $s_k \leq f(x) \leq t_k$ for $x_{k-1} < x < x_k$. Let q_k be a rational number and r_k an irrational number in the open interval (x_{k-1}, x_k), for $k = 1, \ldots, n$. Then $s_k \leq f(r_k) = 0$ and $1 = f(q_k) \leq t_k$ for $k = 1, ..., n$. Therefore, $s(x) \leq 0$ and $t(x) \geq 1$, except possibly at partition points. □

b) The lower integral of f, $\underline{I}(f) = 0$, and the upper integral of f, $\overline{I}(f) = 1$. Hence f cannot be integrable by Theorem 1.9.

Proof. Let s and t be step functions as in part a). Then

$$\int_0^1 s(x)\,dx = \sum_{k=1}^{n} s_k(x_k - x_{k-1}) \leq \sum_{k=1}^{n} 0(x_k - x_{k-1}) = 0$$

and

$$\int_0^1 t(x)\,dx = \sum_{k=1}^{n} t_k(x_k - x_{k-1}) \geq \sum_{k=1}^{n} 1(x_k - x_{k-1}) = 1.$$

We conclude that every number in the set

$$S = \left\{ \int_0^1 s(x)\,dx \;\middle|\; s \leq f \right\}$$

is ≤ 0 and every number in the set

$$T = \left\{ \int_0^1 t(x)\,dx \;\middle|\; f \leq t \right\}$$

is ≥ 1. Therefore $\underline{I}(f) = \sup S \leq 0$ and $\overline{I}(f) = \inf T \geq 1$. Notice also that the constant step functions $s(x) = 0$ and $t(x) = 1$ satisfy $s(x) \leq f(x) \leq t(x)$ for all $0 \leq x \leq 1$, so $0 \in S$ and $1 \in T$. Therefore $\underline{I}(f) = \sup S = 0$ and $\overline{I}(f) = \inf T = 1$. $\square$