MATH 165: HONORS CALCULUS I
ASSIGNMENT 11 SOLUTIONS

Problem 1 Let $A = \left\{1 + \frac{1}{n} - \frac{1}{m} \mid n, m \in \right\}$. Prove that $\sup A = 2$ and $\inf A = 0$.

Proof. We first show $\sup A = 2$. For any $n, m \in$, $\frac{1}{n} \leq 1$ and $-\frac{1}{m} < 0$, so

$$1 + \frac{1}{n} - \frac{1}{m} \leq 1 + 1 - \frac{1}{m} < 2$$

Thus, 2 is an upper bound for A. Since $\sup A$ is the *least* upper bound, it is $\leq$ any other upper bound; in particular, $\sup A \leq 2$. To show $\sup A = 2$, we eliminate the other possibility, $\sup A < 2$, by showing that leads to a contradiction. So suppose $h = \sup A < 2$. Since $2 - \frac{1}{m} \in A$ for all $m \in$, and h is an upper bound for A, we get $2 - \frac{1}{m} \leq h < 2$ for all $m \in$. But this implies that $m \leq \frac{1}{2-h}$ for all $m \in$. We know this is impossible becase is unbounded. In fact, $\left[\frac{1}{2-h}\right] + 1$ is a positive integer that is greater than $\frac{1}{2-h}$. This contradiction implies that $\sup A = 2$.

Now we prove $\inf A = 0$. Since $\frac{1}{n} > 0$ and $-\frac{1}{m} \geq -1$ for all $n, m \in$, we get

$$1 + \frac{1}{n} - \frac{1}{m} \geq 1 + \frac{1}{n} - 1 > 0$$

So, 0 is a lower bound for A. Since $\inf A$ is the *greatest* lower bound, $\inf A \geq 0$. Suppose $k = \inf A > 0$. Then, since $\frac{1}{n} \in A$ for all $n \in$ and k is a lower bound for A, $0 < k \leq \frac{1}{n}$ for all $n \in$. This implies that $n \leq 1/k$ for all $n \in$ which is clearly impossible. In fact, $\left[\frac{1}{k}\right] + 1$ is a positive integer that is greater than $\frac{1}{k}$. This contradiction implies that $\inf A = 0$. □

Problem 2 Define

$$f(x) = \begin{cases} 0 & \text{if } x = 1/n \text{ for some } n \in \\ 1 & \text{otherwise} \end{cases}$$

Prove that $f(x)$ is integrable on $[0, 1]$.

Proof. Fix $n \in$ and define a step function s_n by

$$s_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/n \\ f(x) & \text{if } 1/n < x \leq 1 \end{cases}$$

A partition for s_n is given by the points $x_0 = 0$, $x_1 = 1/n$, $x_2 = 1/(n-1)$, $\ldots$, $x_{n-1} = 1/2$, $x_n = 1$. Define a second step function by $t(x) = 1$ for $0 \le x \le 1$. Then

$$s_n(x) \le f(x) \le t(x) \text{ for all } 0 \le x \le 1$$

The integrals of these steps functions are easy to calculate:

$$\begin{aligned}
\int_0^1 s_n &= \sum_{k=1}^{n} s_k(x_k - x_{k-1}) = 0(x_1 - x_0) + \sum_{k=2}^{n} 1(x_k - x_{k-1}) \\
&= x_n - x_1 = 1 - \frac{1}{n} \\
\int_0^1 t &= 1
\end{aligned}$$

Since $\int_0^1 s \le \underline{I}(f) \le \overline{I}(f) \le \int_0^1 t$ for any step functions satisfying $s(x) \le f(x) \le t(x)$ on $[0, 1]$, we see that

$$1 - \frac{1}{n} \le \underline{I}(f) \le \overline{I}(f) \le 1$$

and these inequlities must hold for any $n \in$. Since $I = 1$ is the only number that satisfies

$$1 - \frac{1}{n} \le I \le 1 \quad \text{for all } n \in$$

we conclude that $\underline{I}(f) = \overline{I}(f) = 1$. Therefore f is integrable and $\int_0^1 f = 1$. $\square$

Problem 3 Use Theorem 1.14 to find an approximation to the integral $\int_1^2 \frac{1}{x}\,dx$ so that the error is $< .05$.

Solution. The function $f(x) = \frac{1}{x}$ is decreasing on $[1, 2]$. The partition points for n subintervals of equal length are given by $x_k = 1 + k\frac{1}{n}$, for $k = 1, \ldots, n$. Let

$$B_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1 + k\frac{1}{n}} = \sum_{k=1}^{n} \frac{1}{n+k}$$

By Theorem 1.14,

$$B_n \le \int_1^2 \frac{1}{x}\,dx \le B_n + E_n$$

where

$$E_n = \frac{(f(1) - f(2))(2-1)}{n} = \frac{1}{2n}$$

If we take our approximation to be the midpoint of the interval $[B_n, B_n + E_n]$, namely $A_n = B_n + E_n/2$, then the true value of the integral will lie with $\pm E_n/2$ of A_n. We would like this error to be less than 0.05:

$$\frac{1}{2}E_n < 0.05 \iff \frac{1}{4n} < 0.05 \iff n > \frac{1}{4 \cdot 0.05} = 5$$

Therefore, the desired approximation is

$$
\begin{aligned}
A_6 &= B_6 + \frac{1}{2}E_6 \\
&= \left(\frac{1}{6+1} + \frac{1}{6+2} + \frac{1}{6+3} + \frac{1}{6+4} + \frac{1}{6+5} + \frac{1}{6+6}\right) + \frac{1}{2}\frac{1}{2\cdot 6} \\
&= 0.694877
\end{aligned}
$$

The actual value of this integral is $\log(2) = 0.693147$ and the error of our approximation is $0.694877 - 0.693147 = 0.00173 < 0.05$.