Math 165: Honors Calculus I Name:______________________________

Exam I *October 1, 1998*

There are 8 problems worth of total of 110 points.

1. (15 pts) Use the axioms for the real numbers[1] to prove the following statements. Be sure to justify each step.

   a) $0 \cdot a = 0$ for all $a \in$ [Hint: consider $0 + 0$]

   b) $(-1)(-1) = 1$ [Hint: consider $1 + (-1)$ and use a)]

   c) $1 > 0$ [Hint: consider a proof by contradiction using b)]

[1] See the Appendix

2. (15 pts)

a) Define an inductive set, S.

b) Define the positive integers, .

c) Prove by induction that for $n \in$,

$$\sum_{k=1}^{n} k(k-1) = \frac{n^3 - n}{3}$$

3. (15 pts)

a) Define completely $\binom{n}{k}$.

b) State the BINOMIAL THEOREM.

c) Prove that for any positive integer n,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k} = 1$$

4. (10 pts)

a) For a subset of $S \subset$, define $\sup S$ and $\inf S$.

b) Let $S = \left\{ \frac{n}{n+1} \mid n \in \right\}$. Prove that $\sup S = 1$.

5. (15 pts) Give precise definitions of the following.

a) $\int_a^b s(x)\,dx$ where s is a step function on $[a,b]$.

b) The lower integral, $\underline{I}(f)$, of a bounded function f on $[a,b]$.

c) An integrable function f on $[a,b]$.

6. (15 pts) Give precise statements of the following theorems for integrals.

a) LINEARITY WITH RESPECT TO THE INTEGRAND.

b) ADDITIVITY WITH RESPECT TO THE INTERVAL OF INTEGRATION.

c) COMPARISON THEOREM

7. (15 pts) Evaluate the following integrals (justify your answers!).

a) $\displaystyle\int_2^3 (x-3)^{16}\,dx.$

b) $\displaystyle\int_{-1}^1 \frac{x}{\sqrt{4-x^4}}\,dx.$

c) Find the area between the graphs of $f(x) = x^3$ and $g(x) = 3x^2 - 2x$ on the interval $[0, 2]$.

8. (10 pts) Use step functions that are constant on subintervals of equal length to compute a numerical approximation for $\int_0^1 \frac{1}{1+x^2}\,dx$ that is accurate to within ± 0.25.

Appendix: Axioms for the Real Numbers

A set , called the set of real numbers, is assumed to exist satisfying the ten axioms below.

The Field Axioms. Two operations on , *addition* and *multiplication*, are assumed to be defined, so that for each pair, x, $y \in$, there is a uniquely determined *sum*, $x+y \in$, and a uniqely determined *product*, $x \cdot y \in$, satisfying the following axioms.

- Axiom 1. $x + y = y + x$, $xy = yx$
- Axiom 2. $x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$
- Axiom 3. $x(y + z) = xy + xz$
- Axiom 4. There exists distinct numbers 0 and 1 such that for every $x \in$, $x + 0 = x$ and $1 \cdot x = x$.
- Axiom 5. For every $x \in$, there is a $y \in$ such that $x + y = 0$.
- Axiom 6. For every $x \in$, $x \neq 0$, there is a $y \in$ such that $xy = 1$.

The Order axioms A subset $^+ \subset$, called the positive numbers, is assumed to exist satisfying the following axioms.

- Axiom 7. If x and y are in $^+$, so are $x + y$ and xy.
- Axiom 8. For every real $x \neq 0$, either $x \in^+$ or $-x \in^+$, but not both.
- Axiom 9. $0 \notin^+$.

The Completeness Axiom

- Axiom 10. For every non-empty subset $S \subset$ that is bounded above there is a $B \in$ that is the supremum of S, $B = \sup S$.