MATH 165: HONORS CALCULUS I
EXAM I SOLUTIONS

(1) Use the axioms for the real numbers to prove the following statements.

a) $0 \cdot a = 0$ for all $a \in$

Proof.

$$\begin{aligned} 0+0 &= 0 && \text{[Axiom 4]} \\ a(0+0) &= a \cdot 0 && \text{[multiply both sides by } a\text{]} \\ a \cdot 0 + a \cdot 0 &= a \cdot 0 && \text{[Axiom 3]} \\ -a \cdot 0 + a \cdot 0 + a \cdot 0 &= -a \cdot 0 + a \cdot 0 && \text{[add } -a \cdot 0 \text{ to both sides]} \\ 0 + a \cdot 0 &= 0 && \text{[Axiom 5]} \\ a \cdot 0 &= 0 && \text{[Axiom 4]} \end{aligned}$$

b) $(-1)(-1) = 1$

Proof.

$$\begin{aligned} 1 + (-1) &= 0 && \text{[Axiom 5]} \\ (-1)(1 + (-1)) &= (-1)0 && \text{[multiply both sides by } (-1)\text{]} \\ (-1)(1 + (-1)) &= 0 && \text{[by a)]} \\ (-1)1 + (-1)(-1) &= 0 && \text{[Axiom 3]} \\ (-1) + (-1)(-1) &= 0 && \text{[Axiom 4]} \\ 1 + (-1) + (-1)(-1) &= 1 + 0 && \text{[add 1 to both sides]} \\ 0 + (-1)(-1) &= 1 && \text{[Axioms 4, 5]} \\ (-1)(-1) &= 1 && \text{[Axiom 5]} \end{aligned}$$

c) $1 > 0$

Proof. If $1 \notin^+$, then $-1 \in^+$ by Axiom 8 ($1 \neq 0$ by Axiom 4). Then Axiom 7 implies that $(-1)(-1) \in^+$. But $(-1)(-1) = 1$ by b), a contradiction. Therefore, $1 \in^+$.

(2) a) Define an inductive set, S.

Definition. A subset $S \subset$ is inductive if
1) $1 \in S$, and
2) if $x \in S$ then $x + 1 \in S$.

b) Define the positive integers, .

Definition. $n \in$ if and only if n is in every inductive set.

c) Prove by induction that for $n \in$,

$$\sum_{k=1}^{n} k(k-1) = \frac{n^3 - n}{3}$$

Proof.
1) The formula holds for $n = 1$:

$$\sum_{k=1}^{1} k(k-1) = 1(1-1) = 0 = \frac{1^3 - 1}{3}$$

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2) Assume the formula holds for n. Then

$$\begin{aligned}\sum_{k=1}^{n+1} k(k-1) &= \sum_{k=1}^{n} k(k-1) + (n+1)n \\ &= \frac{n^3-n}{3} + n^2 + n \quad \text{[induction hypothesis]} \\ &= \frac{n^3 - n + 3n^2 + 3n}{3} \\ &= \frac{(n^3+3n^2+3n+1)-(n+1)}{3} \\ &= \frac{(n+1)^3-(n+1)}{3}\end{aligned}$$

So the formula holds for $n+1$. Therefore, by the Principle of Mathematical Induction, the formula holds for all $n \in$.

(3) a) Define completely $\binom{n}{k}$.

Definition.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

where $n!$ is defined inductively by $0! = 1$ and for $n \geq 1$, $n! = n \cdot (n-1)!$

b) State the Binomial Theorem.

Theorem. For any x, $y \in$ and $n \in$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

c) Prove that for any positive integer n,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} = 1$$

Proof. By the Binomial Theorem,

$$1 = (2-1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^k$$

(4) a) For a subset of $S \subset$, define $\sup S$ and $\inf S$.

Definition.
1) $\sup S$ is the least upper bound for S, i.e., $x \leq \sup S$ for all $x \in S$ and if $x \leq B$ for all $x \in S$, then $\sup S \leq B$.
2) $\inf S$ is the greatest lower for S, i.e., $\inf S \leq x$ for all $x \in S$ and if $B \leq x$ for all $x \in S$, then $B \leq \inf S$.

b) Let $S = \left\{ \frac{n}{n+1} \,\middle|\, n \in \right\}$. Prove that $\sup S = 1$.

Proof. Since $0 < 1$, $n < n+1$ for all $n \in$. Thus, $\frac{n}{n+1} < 1$ for all $n \in$ showing that 1 is an upper bound for S. In particular, $\sup S \leq 1$. Suppose $k = \sup S < 1$. Then $\frac{n}{n+1} \leq k < 1$ for all $n \in$. But then, solving for n and using the fact that $1 - k > 0$, we get $n \leq \frac{k}{1-k}$ for all $n \in$, a contradiction. In fact, $\left[\frac{k}{1-k}\right] + 1$ is a positive integer that is greater than $\frac{k}{1-k}$.

(5) Give precise definitions of the following.

a) $\int_a^b s(x)\,dx$ where s is a step function on $[a,b]$.

Definition. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$ on whose subintervals the step function s is constant:

$$s(x) = s_k \quad \text{for } x_{k-1} < x < x_k, \quad k = 1, \dots, n.$$

Then the integral of s from a to b is defined to be

$$\int_a^b s(x)\,dx = \sum_{k=1}^n s_k(x_k - x_{k-1})$$

b) The lower integral, $\underline{I}(f)$, of a bounded function f on $[a,b]$.

Definition. Let $S = \left\{ \int_a^b s(x)\,dx \;\middle|\; s \le f \right\}$. Here $s \le f$ means that s is a step function on $[a,b]$ such that $s(x) \le f(x)$ for all $x \in [a,b]$. Since f is bounded on $[a,b]$, there is a constant M such that $f(x) \le M$ for $x \in [a,b]$. This implies that $\int_a^b s(x)\,dx \le M(b-a)$ for any step function $s \le f$, so the set S is bounded from above. By Axiom 10, the supremum of S exists and the lower integral is defined to be

$$\underline{I}(f) = \sup S$$

c) An integrable function f on $[a,b]$.

Definition. A function f is integrable on $[a,b]$ if f is bounded on $[a,b]$ and there is *exactly one* number I satisfying

$$\int_a^b s(x)\,dx \le I \le \int_a^b t(x)\,dx$$

for all step functions s, t such that $s(x) \le f(x) \le t(x)$ for $x \in [a,b]$. (This number I is also denoted by $\int_a^b f(x)\,dx$.)

(6) Give precise statements of the following theorems for integrals.

a) Linearity with Respect to the Integrand.

Theorem. If f and g are integrable on $[a,b]$, then for any constants c_1, $c_2 \in$, the function $c_1 f + c_2 g$ is also integrable on $[a,b]$ and

$$\int_a^b c_1 f(x) + c_2 g(x)\,dx = c_1 \int_a^b f(x)\,dx + c_2 \int_a^b g(x)\,dx$$

b) Additivity with Respect to the Interval of Integration.

Theorem. Let $a \le c \le b$. Then f is integrable on $[a,b]$ if and only if f is integrable on $[a,c]$ and on $[c,b]$. Moreover,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

c) Comparison Theorem

Theorem. If f and g are integrable on $[a,b]$ and $f(x) \le g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

(7) a) By Invariance Under Translation,

$$\int_2^3 (x-3)^{16}\,dx = \int_{-1}^0 x^{16}\,dx = \frac{0^{17}}{17} - \frac{(-1)^{17}}{17} = \frac{1}{17}$$

b) $\int_{-1}^1 \frac{x}{\sqrt{4-x^4}}\,dx = 0$. The integral is 0 because $f(x) = \frac{x}{\sqrt{4-x^4}}$ is an odd function, $f(-x) = -f(x)$, and the integral is over a balanced interval, $[-1,1]$.

c) Find the area between the graphs of $f(x) = x^3$ and $g(x) = 3x^2 - 2x$ on the interval $[0, 2]$.

Solution. The area is defined to be

$$A = \int_0^2 |f(x) - g(x)|\, dx$$

Since $f(x) - g(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2)$, we see that $f(x) - g(x)$ is positive on $[0, 1]$ and negative on $[1, 2]$. Therefore,

$$\begin{aligned} A &= \int_0^1 x^3 - 3x^2 + 2x\, dx - \int_1^2 x^3 - 3x^2 + 2x\, dx \\ &= \frac{1}{4}x^4 - x^3 + x^2\Big|_0^1 - \big(\frac{1}{4}x^4 - x^3 + x^2\big)\Big|_1^2 \\ &= \frac{1}{4} - 1 + 1 - \big(4 - \frac{1}{4} - 8 + 1 + 4 - 1\big) \\ &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{aligned}$$

(8) Use step functions that are constant on subintervals of equal length to compute a numerical approximation for $\int_0^1 \frac{1}{1+x^2}\, dx$ that is accurate to within ± 0.25.

Solution. Since $f(x) = \frac{1}{1+x^2}$ is a decreasing function on $[0, 1]$, we know from Theorem 1.14 that

$$B_n \leq \int_0^1 \frac{1}{1+x^2}\, dx \leq B_n + E_n$$

where

$$B_n = \frac{1-0}{n} \sum_{k=1}^{n} \frac{1}{1+x_k^2}$$

and

$$E_n = \frac{(f(0) - f(1))(1-0)}{n} = \frac{1}{2n}$$

The partition points are given by $x_k = \frac{k}{n}$. If we choose the midpoint, $B_n + \frac{1}{2}E_n$, of the interval $[B_n, B_n + E_n]$ for the approximation, then the approximation will be within ± 0.25 of the true value of the integral when $\frac{1}{2}E_n = \frac{1}{4n} < 0.25$. This holds for $n > 1$ (!), so we may take $n = 2$. The midpoint is then

$$\begin{aligned} B_2 + \frac{1}{2}E_2 &= \frac{1}{2}\left(\frac{1}{1+(\frac{1}{2})^2} + \frac{1}{1+1^2}\right) + \frac{1}{8} \\ &= \frac{13}{20} + \frac{1}{8} = 0.65 + 0.125 = 0.775 \end{aligned}$$

(The true value of the integral is $\frac{\pi}{4} \approx 0.785$).