MATH 165: HONORS CALCULUS I
EXAM II SOLUTIONS

(1) a) $\lim_{x\to p} f(x) = A$ means given any $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - p| < \delta$ then $|f(x) - A| < \epsilon$.

b) A function f is continuous at p if f is defined in a neighborhood of p and $\lim_{x\to p} f(x) = f(p)$.

c) The inverse of $f(x)$ is a function $g(x)$ whose domain is the range of f and satisfies: $y = f(x)$ if and only if $g(y) = x$.

d) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

(2) a) Basic Limit Theorems. Let f and g be function such that $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Then

i) $\lim_{x\to p}[f(x) + g(x)] = A + B$,
ii) $\lim_{x\to p}[f(x) - g(x)] = A - B$,
iii) $\lim_{x\to p}[f(x) \cdot g(x)] = A \cdot B$,
iv) $\lim_{x\to p}[f(x)/g(x)] = A/B$, if $B \neq 0$.

b) Intermediate Value Theorem. Let f be continuous on $[a, b]$ and let $x_1 < x_2$ be points in $[a, b]$ such that $f(x_1) \neq f(x_2)$. Then for any value k between $f(x_1)$ and $f(x_2)$ there is at least one point $c \in [x_1, x_2]$ such that $f(p) = c$.

c) Extreme Value Theorem. Let f be continuous on $[a, b]$. Then f has a maximum and a minimum in $[a, b]$, that is, there are points c, $d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a.b]$.

d) Mean Value Theorem for Integrals. Let f be continuous on $[a, b]$. Then f attains its average value at some point in $[a, b]$, that is, there is a $p \in [a, b]$ such that

$$f(p) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

(3) a) $\lim_{x\to 1}(x-1)\sin\dfrac{x}{x-1} = 0$. This follows from the Squeezing Principle: $-1 \leq \sin\dfrac{x}{x-1} \leq 1$ for all $x \neq 1$, so $-|x-1| \leq (x-1)\sin\dfrac{x}{x-1} \leq |x-1|$. Since the outside terms, $\pm|x-1|$, approach 0, the middle term also approaches 0.

b)

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos(x)-1}{x^2} &= \lim_{x\to 0}\frac{(\cos(x)-1)(\cos(x)+1)}{x^2(\cos(x)+1)} = \lim_{x\to 0}\frac{\cos^2(x)-1}{x^2(\cos(x)+1)} \\
&= \lim_{x\to 0}\frac{-\sin^2(x)}{x^2}\frac{1}{\cos(x)+1} = \lim_{x\to 0} -\left[\frac{\sin(x)}{x}\right]^2\frac{1}{\cos(x)+1} \\
&= -1^2\cdot\frac{1}{2} = -\frac{1}{2}
\end{aligned}$$

c)

$$\begin{aligned}
\lim_{x\to 2^-} x^3 - [x^3] &= \lim_{x\to 2^-} x^3 - \lim_{x\to 2^-}[x^3] \quad \text{(both left-sided limits exist)} \\
&= 8 - 7 = 1 \quad (\text{if } 1.95 < x < 2 \text{ then } [x^3] = 7)
\end{aligned}$$

d) Since the right-hand limit, $\lim_{x\to 0^+} x\left|1 + \dfrac{1}{x}\right| = \lim_{x\to 0^+}\dfrac{x}{|x|}|x+1| = 1\cdot|0+1| = 1$, does not equal the left-hand limit, $\lim_{x\to 0^-}\dfrac{x}{|x|}|x+1| = -1\cdot|0+1| = -1$, the limit $\lim_{x\to 0} x\left|1+\dfrac{1}{x}\right|$ does not exist.

Date: November 19, 1998.

(4) a) Let $f(x) = x^{2/3} + x^{-1}$, $x > 0$. Then, by the power rule, $f'(x) = (2/3)x^{2/3-1} + (-1)x^{-1-1} = (2/3)x^{-1/3} - x^{-2}$.

b) Let a and b be constants, not both zero, and let $f(x) = \dfrac{ax+b}{a-bx}$. Then, by the quotient rule, $f'(x) = \dfrac{a(a-bx)-(-b)(ax+b)}{(a-bx)^2} = \dfrac{a^2+b^2}{(a-bx)^2} > 0$ for all $x \neq a/b$.

c) Let $f(x) = (ax+b)\cos(x) + (cx+d)\sin(x)$. Determine values of the constants a, b, c, d, such that $f'(x) = x\sin(x)$.

Solution. By the power rule, $f'(x) = a\cos(x) - (ax+b)\sin(x) + c\sin(x) + (cx+d)\cos(x) = (cx+d+a)\cos(x) + (-ax-b+c)\sin(x)$. This equals $x\sin(x)$ when $-a = 1$, $-b+c = 0$, $c = 0$, and $d+a = 0$, i.e., when $a = -1$, $b = c = 0$, and $d = 1$. Thus $f(x) = -x\cos(x) + \sin(x)$.

(5) Let f be an integrable function on $[a,b]$ and let $F(x) = \displaystyle\int_a^x f(t)\,dt$. Prove that F is continuous at $c \in [a,b]$.

Proof. It is enough to show that $\lim_{x\to c} F(x) - F(c) = 0$. First note that

$$F(x) - F(c) = \int_a^x f(x)\,dx - \int_a^c f(x)\,dx = \int_c^x f(x)\,dx$$

Since f is integrable on $[a,b]$, it is bounded on $[a,b]$ by definition, so there is a constant $M > 0$ such that $-M \le f(x) \le M$ for all $x \in [a,b]$. This implies that $-M|x-c| \le \displaystyle\int_c^x f(x)\,dx \le M|x-c|$, so $|F(x)-F(c)| < M|x-c|$. Given any $\epsilon > 0$, let $\delta = \epsilon/M$. If $|x-c| < \delta$, then $|F(x)-F(c)| < M|x-c| < M\delta = \epsilon$.

(6) Let $n \ge 2$ be a positive integer. Prove that the polynomial $f(x) = x^n - nx + 1$ has a root in $[0,1]$.

Proof. If $n = 2$, then $f(x) = x^2 - 2x + 1$ and $f(1) = 0$. So we may assume $n > 2$. Since $f(0) = 1$ and $f(1) = 1 - n + 1 < 0$, Balzano's theorem implies that there is a $c \in [0,1]$ such that $f(c) = 0$.

(7) Find the largest interval I containing $x = 1$ on which the function $f(x) = \dfrac{1}{1+x^2}$ has an inverse. Give a formula for the corresponding inverse function, $f^{-1}(x)$, as a function of x and determine its domain.

Solution. First note that f is strictly decreasing for $x \ge 0$: If $0 \le x_1 < x_2$ then $1 + x_1^2 < 1 + x_2^2$ so $f(x_2) = \dfrac{1}{1+x_2} < \dfrac{1}{1+x_1^2} = f(x_1)$. Therefore, f is 1-1 on $[0,\infty)$. Now f cannot be 1-1 on a larger interval, since any such interval would contain a negative number, say $-a < 0$, and $f(-a) = f(a)$ showing that f is not 1-1 on that interval.

To find $f^{-1}(x)$ we solve the equation $y = \dfrac{1}{1+x^2}$ for x to get $x = +\sqrt{\dfrac{1}{y} - 1}$. The range of f on the interval $[0,\infty)$ is $(0,1]$: it is clear that $0 < \dfrac{1}{1+x^2} \le 1$ and any number $0 < y \le 1$ is achieved by $f(x)$ when $x = \sqrt{\dfrac{1}{y} - 1}$. Thus the inverse function is $f^{-1}(x) = \sqrt{\dfrac{1}{x} - 1}$ and its domain is $(0,1]$.

(8) a) Using the definition of continuity, prove that $f(x) = x^2$ is continuous at any real number p.

Proof. We must show that given any $\epsilon > 0$ there is a $\delta > 0$ such that if $|x-p| < \delta$ then $|f(x)-f(p)| < \epsilon$. If x is within 1 unit of p, i.e. if $|x-p| < 1$, then $|x| < |p|+1$, so $|x+p| < |x|+|p| < 2|p|+1$. This allows us to estimate $|f(x)-f(p)|$ in terms of $|x-p|$: $|f(x)-f(p)| = |x^2-p^2| = |x+p||x-p| < (2|p|+1)|x-p|$. Let $\delta = \min\{1, \epsilon/(2|p|+1)\}$. Then $|x-p| < \delta$ implies both $|x-p| < 1$ and $|x-p| < \epsilon/(2|p|+1)$ and we get $|f(x)-f(p)| < (2|p|+1)|x-p| < (2|p|+1)\epsilon/(2|p|+1) = \epsilon$.

b) Using the definition of the derivative, prove that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \dfrac{1}{2\sqrt{x}}$.

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$