Math 165: Honors Calculus I Name:________________________

Final Exam *Dec. 15, 1998*

There are 12 pages of problems worth a total of 165 points which includes 15 bonus points.

1. (30 pts) Give complete definitions for the following concepts. Define any symbols used.

   a) The natural numbers $\mathbf{N}$

   b) $\sum_{i=1}^{n} a_i$

   c) A function f is integrable on $[a, b]$

d) $\lim_{x \to p} f(x) = A$

e) A function f is differentiable at p

f) A function f is convex on $[a, b]$.

2. (30 pts) State the following theorems precisely. Define any new symbols or concepts used.

   a) Binomial Theorem

   b) The Squeezing Principle

   c) Boundedness Theorem

d) Chain Rule

e) Mean Value Theorem for Derivatives

f) Second Derivative Test for Extrema

3. (20 pts) Calculate the following.

a) $\int_0^{16} [\sqrt{x}]\, dx$ where $[u]$ is the greatest integer $\leq u$.

b) $\int_0^2 |x^2 - 1|\, dx$

c) The average value of $\sin(x)$ on the interval $[0, \pi]$.

d) $\int_{-1}^{1} \frac{\tan(x)}{x^2+1}\, dx$

4. (20 pts) Compute the following limits (do not use L'Hopitâl's Rule) or prove they do not exist.

a) $\displaystyle\lim_{x\to 0} \frac{\sin(x^2) + \sin^2(x)}{x^2}$

b) $\displaystyle\lim_{x\to 1^-} \frac{\sqrt{x^3 - 2x^2 + x}}{x-1}$

c) $\displaystyle\lim_{x\to 0}(x^2 - x)\cos\left(\frac{1}{x}\right)$

d) $\displaystyle\lim_{x\to a} \frac{(x^2+1)^{100} - (a^2+1)^{100}}{x-a}$

5. (15 pts) Compute the derivative, $f'(x)$, of the function $f(x)$.

a) $f(x) = x^3 \cos(x)$

b) $f(x) = \dfrac{x^2 + x + 1}{x^2 - x + 1}$

c) $f(x) = \sqrt{1 + \sqrt{1 + \sqrt{x}}}$

6. (5 pts) Find the equation of the line tangent to the curve defined by

$$y^3 + x^2y^2 - x = 1$$

at the point $(1, 1)$.

7. (5 pts) Consider the sequence $a_0 = 1,\ a_1 = 1,\ a_2 = 10,\ a_3 = 28,\ \ldots,$ defined recursively by

$$a_{n+1} = 2a_n + 8a_{n-1}, \quad n \geq 1$$

For example, $a_4 = 2a_3 + 8a_2 = 2(28) + 8(10) = 136$. Use the PRINCIPLE OF MATHEMATICAL INDUCTION to prove that

$$a_n = \frac{4^n + (-2)^n}{2}, \quad n \geq 1$$

8. (5 pts) Use the *definition of a limit* to prove that $\lim_{x \to 1} \frac{1}{x} = 1$.

9. (5 pts) Prove that if a function f is differentiable at c and has a relative maximum at c then $f'(c) = 0$.

10. (5 pts) Let $n \geq 3$ be a positive integer. Use Bolzano's Theorem and Rolle's Theorem to prove that the function $f(x) = x^n - nx + 1$ has *exactly one* root in the interval $(0, 1)$.

11. (5 pts) Prove that if $f(x)$ is a strictly increasing function, then it has an inverse, $f^{-1}(x)$ that is also strictly increasing.

12. (5 pts) Prove that $\sup\left\{ \dfrac{x-1}{x+1} \mid x \geq 0 \right\} = 1$

13. (5 pts) Find the height x and circumference y of a right circular cylinder of maximum volume that can be legally sent through the US mail ($x + y = 108$ in).

14. (10 pts) Let $f(x) = \frac{3}{5}x^5 - 5x^3 + 12x - 3$.

a) Determine the intervals on which f is increasing and decreasing.

b) Determine the relative extrema of f.

c) Determine the intervals on which f is convex and concave.

d) Sketch the graph of f.