Math 165: Honors Calculus I Name:__________________________
Exam I *September 30, 1999*

There are 8 problems worth of total of 110 points.

(1) (15 pts) Justify your statements using the axioms for the real numbers (see the Appendix) or previous parts of the problem. Let $a \in \mathbb{R}$ be an arbitrary real number.
a) Define $-a$.

b) Prove that $0 \cdot a = 0$. [Hint: $0 + 0 = 0$]

c) Prove that $(-1) \cdot a = -a$. [Hint: $1 + (-1) = 0$]

(2) (15 pts)

a) Define the positive integers, $\mathbb{N}$.

b) Prove that there is no upper bound for $\mathbb{N}$.

c) Let $a > 0$ and let $T = \left\{ a\left(1 + \frac{1}{n}\right) \mid n \in \mathbb{N} \right\}$. Prove that $a = \inf(T)$.

(3) (15 pts)

a) Define $\sum_{k=1}^{n} a_k$.

b) Use induction to prove that for any positive integer n, $\sum_{k=1}^{n} (k-1)^3 < \frac{n^4}{4}$.

c) Use telescoping sums to calculate $\sum_{k=1}^{n} \frac{2k-1}{2^k k!}$.

(4) (10 pts) Find the coefficient of x^{81} in $(x^3 - 7)^{30}$ (give its prime factorization).

(5) (20 pts) Give precise definitions of the following.

a) A step function s on $[a, b]$.

b) $\displaystyle\int_a^b s(x)\,dx$ where s is a step function on $[a, b]$.

c) The lower integral, $\underline{I}(f)$, and upper integral, $\overline{I}(f)$, of a bounded function f on $[a, b]$.

d) $\displaystyle\int_a^b f(x)\,dx$ where f is a bounded function on $[a, b]$.

(6) (15 pts) Give precise statements of the following theorems for integrals.

a) LINEARITY WITH RESPECT TO THE INTEGRAND.

b) ADDITIVITY WITH RESPECT TO THE INTERVAL OF INTEGRATION.

c) COMPARISON THEOREM.

(7) (10 pts) Evaluate $\displaystyle\int_{-2}^{2} 3 - |x^2 - 1|\,dx$.

(8) (10 pts) Let $b > 0$. Show that $\int_0^b x\,dx = \frac{1}{2}b^2$ by considering step functions constant on subintervals of equal length.

[Hint: You may find the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ useful.]

Appendix: Axioms for the Real Numbers A set $\mathbb{R}$, called the set of real numbers, is assumed to exist satisfying the ten axioms below.

The Field Axioms. Two operations on $\mathbb{R}$, *addition* and *multiplication*, are assumed to be defined, so that for each pair, x, $y \in \mathbb{R}$, there is a uniquely determined *sum*, $x + y \in \mathbb{R}$, and a uniquely determined *product*, $x \cdot y \in \mathbb{R}$, satisfying the following axioms.

- Axiom 1. $x + y = y + x$, $xy = yx$
- Axiom 2. $x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$
- Axiom 3. $x(y + z) = xy + xz$
- Axiom 4. There exists distinct numbers 0 and 1 such that for every $x \in \mathbb{R}$, $x + 0 = x$ and $1 \cdot x = x$.
- Axiom 5. For every $x \in \mathbb{R}$, there is a $y \in \mathbb{R}$ such that $x + y = 0$.
- Axiom 6. For every $x \in \mathbb{R}$, $x \neq 0$, there is a $y \in \mathbb{R}$ such that $xy = 1$.

The Order axioms A subset $\mathbb{R}^+ \subset \mathbb{R}$, called the positive numbers, is assumed to exist satisfying the following axioms.

- Axiom 7. If x and y are in $\mathbb{R}^+$, so are $x + y$ and xy.
- Axiom 8. For every real $x \neq 0$, either $x \in \mathbb{R}^+$ or $-x \in \mathbb{R}^+$, but not both.
- Axiom 9. $0 \notin \mathbb{R}^+$.

The Completeness Axiom

- Axiom 10. For every non-empty subset $S \subset \mathbb{R}$ that is bounded above there is a $B \in \mathbb{R}$ that is the supremum of S, $B = \sup S$.