MATH 165: HONORS CALCULUS I
EXAM I SOLUTIONS

(1) Let $a \in \mathbb{R}$ be an arbitrary real number.

a) Define $-a$.

Definition. $-a$ is the unique number x such that $a + x = 0$. It exists by Axiom 5. It is unique because if x' is another such number then $a + x = 0 = a + x'$ and adding x to both sides gives $(x + a) + x = (x + a) + x'$, which simplifies to $0 + x = 0 + x'$ and $x = x'$ using Axioms 1, 2 and 4.

b) Prove that $0 \cdot a = 0$.

Proof. By Axiom 4, $0 + 0 = 0$, multiplying by a gives $(0 + 0)a = 0 \cdot a$, and by Axiom 3, $0 \cdot a + 0 \cdot a = 0 \cdot a$. Now adding $-(0 \cdot a)$ to both sides gives $-(0 \cdot a) + (0 \cdot a + 0 \cdot a) = -(0 \cdot a) + 0 \cdot a$ which simplifies to $0 + 0 \cdot a = 0$ and $0 \cdot a = 0$ using Axioms 1, 2 and 4, and the definition of $-(0 \cdot a)$.

c) Prove that $(-1) \cdot a = -a$.

Proof. By definition of -1, $1 + (-1) = 0$. Multiplying by a gives $(1 + (-1))a = 0 \cdot a$, and by Axiom 3, $1 \cdot a + (-1) \cdot a = 0 \cdot a$. Using part b) and Axiom 4, this simplifies to $a + (-1) \cdot a = 0$. Since $-a$ is the unique number x satisfying $a + x = 0$ (see part a)), we conclude that $(-1) \cdot a = -a$.

(2) a) Define the positive integers, $\mathbb{N}$.

Definition. $n \in \mathbb{N}$ if and only if n is in every inductive set. A set S is inductive if $1 \in S$, and $x + 1 \in S$ whenever $x \in S$.

b) Prove that there is no upper bound for $\mathbb{N}$.

Proof. If $\mathbb{N}$ has an upper bound, then it has a least upper bound $b = \sup(\mathbb{N})$ by Axiom 10. Since $b - 1$ cannot be an upper bound of $\mathbb{N}$, there must be some $n \in \mathbb{N}$ such that $b - 1 < n$. But then $n + 1 \in \mathbb{N}$ and $b < n + 1$, contradicting the fact that b is an upper bound of $\mathbb{N}$. Therefore, there can be no upper bound of $\mathbb{N}$.

c) Let $a > 0$ and let $T = \left\{a\left(1 + \frac{1}{n}\right) \mid n \in \mathbb{N}\right\}$. Prove that $a = \inf(T)$.

Proof. Since $a/n > 0$ for all $n \in \mathbb{N}$, $a < a + a/n$. Thus, a is a lower bound of T. Suppose b is another lower bound of T and $b > a$. Then $a < b \le a + a/n$ for all $n \in \mathbb{N}$. This implies that $0 < b - a \le a/n$ and $n \le a/(b - a)$ for all $n \in \mathbb{N}$. But then $a/(b - a)$ is an upper bound for $\mathbb{N}$ contradicting part b). Therefore, $b \le a$ and a must be the greatest lower bound of T, $a = \inf(T)$.

(3) a) Define $\displaystyle\sum_{k=1}^{n} a_k$.

Definition. $\displaystyle\sum_{k=1}^{1} a_k$ is defined to be a_1 and if $\displaystyle\sum_{k=1}^{n} a_k$ has been defined for a positive integer n, then $\displaystyle\sum_{k=1}^{n+1} a_k$ is defined to be $\left(\displaystyle\sum_{k=1}^{n} a_k\right) + a_{n+1}$.

b) Use induction to prove that for any positive integer n, $\displaystyle\sum_{k=1}^{n}(k-1)^3 < \frac{n^4}{4}$.

Proof.

i) The inequality holds for $n = 1$ since $\displaystyle\sum_{k=1}^{1}(k-1)^3 = (1-1)^3 = 0 < \frac{1^4}{4}$.

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ii) Assume the inequality holds for n. Then

$$\begin{aligned}\sum_{k=1}^{n+1}(k-1)^3 &= \Big(\sum_{k=1}^{n}(k-1)^3\Big)+n^3 < \frac{n^4}{4}+n^3 = \frac{1}{4}(n^4+4n^3)\\ &< \frac{1}{4}(n^4+4n^3+6n^2+4n+1) = \frac{(n+1)^4}{4}\end{aligned}$$

c) Use telescoping sums to calculate $\displaystyle\sum_{k=1}^{n}\frac{2k-1}{2^k k!}$

Solution.

$$\begin{aligned}\sum_{k=1}^{n}\frac{2k-1}{2^k k!} &= \sum_{k=1}^{n}\frac{2k}{2^k k!}-\frac{1}{2^k k!} = \sum_{k=1}^{n}\frac{1}{2^{k-1}(k-1)!}-\frac{1}{2^k k!}\\ &= \frac{1}{2^0 0!}-\frac{1}{2^n n!} = 1-\frac{1}{2^n n!}\end{aligned}$$

(4) Find the coefficient of x^{81} in $(x^3-7)^{30}$ (give its prime factorization).

Solution. By the binomial theorem, the terms in the expansion of $(x^3-7)^{30}$ have the form $\binom{30}{k}(x^3)^k(-7)^{30-k}$. The term containing x^{81} corresponds to $k=27$. Therefore, the coefficient of x^{81} is

$$\binom{30}{27}(-7)^3 = \frac{30\cdot 29\cdot 28}{3\cdot 2\cdot 1}(-7)^3 = -4\cdot 5\cdot 7^4\cdot 29$$

(5) Give precise definitions of the following.

a) A step function s on $[a,b]$.

Definition. A function $s(x)$ is a step function on $[a,b]$ if there is a partition $P=\{x_0,x_1,\ldots,x_n\}$ of $[a,b]$ such that $s(x)$ is constant on the open subintervals of P: $s(x)=s_k$ for $x\in(x_{k-1},x_k)$, $k=1,\ldots,n$.

b) $\displaystyle\int_a^b s(x)\,dx$ where s is a step function on $[a,b]$.

Definition. Using the notation of part a), $\displaystyle\int_a^b s(x)\,dx$ is defined to be $\displaystyle\sum_{k=1}^{n}s_k(x_k-x_{k-1})$.

c) The lower integral, $\underline{I}(f)$, and upper integral, $\overline{I}(f)$, of a bounded function f on $[a,b]$.

Definition. Let S be the set of numbers $\displaystyle\int_a^b s(x)\,dx$ where s runs through all the step functions on $[a,b]$ below f and let T be the set of numbers $\displaystyle\int_a^b t(x)\,dx$ where t runs through all the step functions on $[a,b]$ above f. Since f is bounded on $[a,b]$, the sets S and T are non-empty and bounded from above and below, respectively. By Axiom 10, the supremum of S and the infimum of T exist. The lower integral of f is defined to be $\underline{I}(f)=\sup(S)$, and the upper integral of f is defined to be $\underline{I}(f)=\inf(T)$.

d) $\displaystyle\int_a^b f(x)\,dx$ where f is a bounded function on $[a,b]$.

Definition. If there is *exactly one* number I satisfying $\displaystyle\int_a^b s(x)\,dx\le I\le\int_a^b t(x)\,dx$ for all step functions s, t such that $s(x)\le f(x)\le t(x)$ for $x\in[a,b]$, then $\displaystyle\int_a^b f(x)\,dx$ is defined to be this number I. Alternately, if $\underline{I}(f)=\overline{I}(f)=I$, where $\underline{I}(f)$ and $\overline{I}(f)$ are defined as in part c), then $\displaystyle\int_a^b f(x)\,dx = I$.

(6) Give precise statements of the following theorems for integrals.

a) Linearity with Respect to the Integrand.

Theorem. If f and g are integrable on $[a,b]$, then for any constants $c_1,\ c_2 \in \mathbb{R}$, the function $c_1 f + c_2 g$ is also integrable on $[a,b]$ and

$$\int_a^b c_1 f(x) + c_2 g(x)\,dx = c_1 \int_a^b f(x)\,dx + c_2 \int_a^b g(x)\,dx$$

b) Additivity with Respect to the Interval of Integration.

Theorem. Let $a \le c \le b$. Then f is integrable on $[a,b]$ if and only if f is integrable on $[a,c]$ and on $[c,b]$. Moreover,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

c) Comparison Theorem

Theorem. If f and g are integrable on $[a,b]$ and $f(x) \le g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

(7) Evaluate $\displaystyle\int_{-2}^{2} 3 - |x^2 - 1|\,dx$.

Solution. $x^2 - 1 \ge 0$ if and only if $x \le -1$ or $x \ge 1$. Using additivity with respect to the interval, the definition of absolute value, and the formula $\displaystyle\int_a^b x^n\,dx = (b^{n+1} - a^{n+1})/(n+1)$, we get

$$\begin{aligned}\int_{-2}^{2} 3 - |x^2 - 1|\,dx &= \int_{-2}^{-1} 3 - (x^2 - 1)\,dx + \int_{-1}^{1} 3 + (x^2 - 1)\,dx + \int_{1}^{2} 3 - (x^2 - 1)\,dx \\ &= (4 - \frac{1}{3}7) + (4 + \frac{1}{3}2) + (4 - \frac{1}{3}7) = 8\end{aligned}$$

(8) Let $b > 0$. Show that $\displaystyle\int_0^b x\,dx = \frac{1}{2}b^2$ by considering step functions constant on subintervals of equal length.

Solution. The partition points for n subintervals of equal length are given by $x_k = k\dfrac{b}{n}$, $k = 0, \ldots, n$. Define step functions by

$$\begin{aligned} s(x) &= s_k = (k-1)\frac{b}{n}, && x_{k-1} \le x < x_k \\ t(x) &= t_k = k\frac{b}{n}, && x_{k-1} < x \le x_k \end{aligned}$$

for $k = 1, \ldots, n$ (also define $s(b) = b$ and $t(0) = 0$). Since $f(x) = x$ is increasing on $[0,b]$, $s(x) \le f(x) \le t(x)$ for all $x \in [0,b]$.

The integrals of these step functions can be easily calculated since $(x_k - x_{k-1}) = \dfrac{b}{n}$:

$$\int_0^b s(x)\,dx = \sum_{k=1}^{n} s_k(x_k - x_{k-1}) = \sum_{k=1}^{n} (k-1)\frac{b}{n}\Big(\frac{b}{n}\Big) = \frac{b^2}{n^2}\sum_{k=1}^{n}(k-1) = \frac{b^2}{n^2}\frac{n(n-1)}{2} = \frac{1}{2}b^2\Big(1 - \frac{1}{n}\Big)$$

Similarly,

$$\int_0^b t(x)\,dx = \sum_{k=1}^{n} t_k(x_k - x_{k-1}) = \sum_{k=1}^{n} k\frac{b}{n}\Big(\frac{b}{n}\Big) = \frac{b^2}{n^2}\sum_{k=1}^{n} k = \frac{b^2}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}b^2\Big(1 + \frac{1}{n}\Big)$$

There is only one number I that satisfies

$$\frac{1}{2}b^2\Big(1 - \frac{1}{n}\Big) \le I \le \frac{1}{2}b^2\Big(1 + \frac{1}{n}\Big)$$

for all $n \in \mathbb{N}$, namely, $I = \dfrac{1}{2}b^2$ (see for example, Problem 2c). We conclude that $\displaystyle\int_0^1 x\,dx = \frac{1}{2}b^2$.