MATH 165: HONORS CALCULUS I
EXAM II SOLUTIONS

(1) a) The area between the graphs of $f(x)$ and $g(x)$ on $[a, b]$ is defined to be $\int_a^b |f(x) - g(x)|\, dx$.

b) Consider a unit circle centered at the origin $O = (0, 0)$ and let A be the point $(1, 0)$. Given a number x between 0 and 2π, let P_x be the point on the unit circle such that AOP_x forms a circular sector with area $x/2$. Then $\cos(x)$ and $\sin(x)$ are the coordinates of $P_x = (\cos(x), \sin(x))$. The domain of $\sin(x)$ and $\cos(x)$ is extended to all real numbers by periodicity: $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$.

c) $\lim_{x\to p} f(x) = A$ means that for any $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - p| < \delta$ then $|f(x) - A| < \epsilon$.

d) A function $f(x)$ is one-to-one if for any two distinct numbers $x_1 \neq x_2$ in its domain, the function values are distinct, $f(x_1) \neq f(x_2)$.

e) The average value of an integrable function on $[a, b]$ is $\overline{f} = \frac{1}{b-a}\int_a^b f(x)\, dx$.

(2) a) Basic Limit Theorems. Let f and g be function such that $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Then

i) $\lim_{x\to p}[f(x) + g(x)] = A + B$,

ii) $\lim_{x\to p}[f(x) - g(x)] = A - B$,

iii) $\lim_{x\to p}[f(x) \cdot g(x)] = A \cdot B$,

iv) $\lim_{x\to p}[f(x)/g(x)] = A/B$, if $B \neq 0$.

b) The Squeezing Principle. If $g(x) \leq f(x) \leq h(x)$ for x in a neighborhood of p ($x \neq p$), and if $\lim_{x\to p} g(x) = A = \lim_{x\to p} h(x)$, then $\lim_{x\to p} f(x) = A$.

c) Bolzano's Theorem. Let f be continuous on $[a, b]$ and assume $f(a)$ and $f(b)$ have opposite signs. Then there is a $c \in [a, b]$ such that $f(c) = 0$.

d) Intermediate Value Theorem. Let f be continuous on $[a, b]$ and let $x_1 < x_2$ be points in $[a, b]$ such that $f(x_1) \neq f(x_2)$. Then for any value k between $f(x_1)$ and $f(x_2)$ there is at least one point $c \in [x_1, x_2]$ such that $f(p) = c$.

e) Extreme Value Theorem. Let f be continuous on $[a, b]$. Then f has a maximum and a minimum in $[a, b]$, that is, there are points c, $d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a.b]$.

(3) a) $\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1} = \lim_{x\to 1}(x+1)\frac{\sin(x^2-1)}{x^2-1} = \lim_{x\to 1}(x+1)\lim_{x\to 1}\frac{\sin(x^2-1)}{x^2-1} = 2 \cdot 1 = 2$

b) $\lim_{x\to 0} \frac{\sqrt{x^4+1}-1}{x^4} = \lim_{x\to 0} \frac{(x^4+1)-1}{x^4(\sqrt{x^4+1}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{x^4+1}+1} = \frac{1}{2}$

c) $\lim_{x\to 2} \sqrt{x^3 - \frac{1}{x^3}} = \sqrt{2^3 - \frac{1}{2^3}} = \sqrt{\frac{63}{8}}$

Date: November 11, 1999.

d) Since $\left[\frac{1}{x}\right] \leq \frac{1}{x}$, we get $0 \leq x^2\Big(1+\left[\frac{1}{x}\right]\Big) \leq x^2\Big(1+\frac{1}{x}\Big) = x^2+x$ for $x > 0$. Since $\lim\limits_{x\to 0^+} x^2+x = 0$, the SQUEEZING PRINCIPLE implies that $\lim\limits_{x\to 0^+} x^2\Big(1+\left[\frac{1}{x}\right]\Big) = 0$

(4) Assume $\lim\limits_{x\to p} f(x) = A$ and $\lim\limits_{x\to p} g(x) = B$. Using the definition of limit, prove that $\lim\limits_{x\to p}\big(f(x)-g(x)\big) = A-B$.
Proof. Let $\epsilon > 0$. By assumption, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that if $0 < |x-p| < \delta_1$ then $|f(x)-A| < \epsilon/2$, and if $0 < |x-p| < \delta_2$ then $|g(x)-B| < \epsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-p| < \delta$ then $|(f(x)-g(x))-(A-B)| \leq |f(x)-A| + |g(x)-B| < \epsilon/2+\epsilon/2 = \epsilon$. Therefore, $\lim\limits_{x\to p}\big(f(x)-g(x)\big) = A-B$.

(5) Let f be continuous at p and let g be continuous at $q = f(p)$. Prove that $g \circ f$ is continuous at p.
Proof. Let $\epsilon > 0$. By assumption, there is a $\delta_1 > 0$ such that if $0 < |y-q| < \delta_1$ then $|g(y)-g(q)| < \epsilon$. Also by assumption, there is a $\delta_2 > 0$ such that if $0 < |x-p| < \delta_2$ then $|f(x)-f(p)| < \delta_1$. By what we've just written, this implies that $|g(f(x))-g(f(p))| < \epsilon$. Therefore, $g(f(x))$ is continuous at $x = p$.

(6) Let f be integrable and positive on $[a,b]$. Show that the function $F(x) = \int_a^x f(t)\,dt$ is increasing on $[a,b]$.
Proof. Let $a \leq x_1 < x_2 \leq b$. Then $F(x_2)-F(x_1) = \int_a^{x_2} f(t)\,dt - \int_a^{x_1} f(t)\,dt = \int_{x_1}^{x_2} f(t)\,dt$. Since $f(t) > 0$ for $t \in [a,b]$, $\int_{x_1}^{x_2} f(t)\,dt \geq \int_{x_1}^{x_2} 0\,dt = 0$ by the COMPARISON THEOREM. Therefore, $F(x_2)-F(x_1) \geq 0$ and hence $F(x)$ is increasing on $[a,b]$.

(7) Suppose the inverse of a function is $f^{-1}(x) = \sqrt{\frac{1}{\sqrt{x}}-1}$ for $0 < x \leq 1$. Determine the original function, $f(x)$, including its domain.
Solution. By definition, $y = f^{-1}(x)$ if and only if $f(y) = x$. To find $f(x)$ we solve the equation $y = \sqrt{\frac{1}{\sqrt{x}}-1}$ for x to get $x = \frac{1}{(y^2+1)^2} = f(y)$, or, $f(x) = \frac{1}{(x^2+1)^2}$. The range of f^{-1} on the interval $(0,1]$ is $[0,\infty)$ and this becomes the domain of f.