Math 165: Honors Calculus I Name:________________________

Final Exam *December 15, 1999*

There are 10 problems on 12 pages worth a total of 165 points.

1. (40 pts) Give complete definitions for the following concepts. Define any symbols used.

   (a) The natural numbers $\mathbf{N}$

   (b) $\sum_{i=1}^{n} a_i$

   (c) The supremum, $\sup S$, of a subset of real numbers $S \subset \mathbf{R}$.

   (d) A function f is integrable on $[a, b]$

(e) $\lim_{x \to p} f(x) = A$

(f) A function f is continuous at p

(g) A function f is differentiable at p

(h) A function f is convex on $[a, b]$.

2. (40 pts) State the following theorems precisely. Define any new symbols or concepts used.

   (a) Binomial Theorem

   (b) The Linearity Property of Integrals

   (c) Sign-Preserving Property of Continuous Functions

   (d) Intermediate Value Theorem

(e) Extreme Value Theorem

(f) Mean Value Theorem for Integrals

(g) Mean Value Theorem for Derivatives

(h) First Derivative Test for Maxima and Minima

3. (15 pts) Calculate the following.

(a) $\int_0^2 [x^2]\, dx$ where $[u]$ is the greatest integer $\leq u$.

(b) Find the area between the graphs of $\sin(x)$ and $\cos(x)$ on $[0, 2\pi]$.

(c) The average value of $f(x) = x(x-1)$ on the interval $[0, 2]$.

4. (10 pts) Compute the following limits (do not use L'Hôpital's Rule!) or prove they do not exist.

(a) $\lim_{x \to 1} \frac{x}{x-1} \sqrt{1 - \frac{2}{x} + \frac{1}{x^2}}$

(b) $\lim_{x \to 0} \sin(x) \cos(1/x)$

5. (10 pts)

(a) Find the maximum and the minimum of $f(x) = \dfrac{x^2+1}{x^3+2}$ on $[0, 2]$.

(b) Find the line tangent to the ellipse $x^2 + 4y^2 = 4$ at the point $(\sqrt{3}, -1/2)$.

6. (10 pts) Use the definition of the derivative to prove that

$$\frac{d}{dx}\sin(x) = \cos(x)$$

You may use trigonometric identities and the fact that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$.

7. (10 pts) Use the definition of the limit to prove that $\lim_{x \to 3} x^2 = 9$.

8. (10 pts) Use the MEAN VALUE THEOREM to prove that if $f'(x) > 0$ on an interval (a, b), then $f(x)$ is strictly increasing on that interval.

9. (10 pts)

(a) Show that if $f(x)$ is a polynomial of degree n, then $f'(x)$ is a polynomial of degree $n-1$.

(b) Use Rolle's Theorem, induction, and part a) to prove that if $f(x)$ is a polynomial of degree n then $f(x)$ has at most n distinct roots.

10. (10 pts) Let $f(x) = \frac{2}{5}x^5 - 3x^3 + 7x$.

(a) Determine the intervals on which f is increasing and decreasing.

(b) Determine the relative maxima and minima of f.

(c) Determine the intervals on which f is convex and concave.

(d) Sketch the graph of f.