MATH 165: HONORS CALCULUS I
FINAL EXAM SOLUTIONS

(1) Give complete definitions...
See the text.

(2) State the following theorems...
See the text.

(3) Calculate the following.

(a) $\int_0^2 [x^2]\,dx$ where $[u]$ is the greatest integer $\le u$.

Solution.

$$\begin{aligned}\int_0^2 [x^2]\,dx &= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^2 3\,dx \\ &= (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3}) = 5-\sqrt{3}-\sqrt{2}\end{aligned}$$

(b) Find the area between the graphs of $\sin(x)$ and $\cos(x)$ on $[0, 2\pi]$.

Solution.

$$\begin{aligned}A &= \int_0^{2\pi} |\sin(x)-\cos(x)|\,dx \\ &= \int_0^{\pi/4} \cos(x)-\sin(x)\,dx + \int_{\pi/4}^{5\pi/4} \sin(x)-\cos(x)\,dx + \int_{5\pi/4}^{2\pi} \cos(x)-\sin(x)\,dx \\ &= (\sin(x)+\cos(x))\Big|_0^{\pi/4} + (-\cos(x)-\sin(x))\Big|_{\pi/4}^{5\pi/4} + (\sin(x)+\cos(x))\Big|_{5\pi/4}^{2\pi} = 4\sqrt{2}\end{aligned}$$

(c) The average value of $f(x) = x(x-1)$ on the interval $[0, 2]$.

Solution. $\overline{f} = \frac{1}{2}\int_0^2 x^2 - x\,dx = \frac{1}{2}(\frac{1}{3}x^3 - \frac{1}{2}x^2)\Big|_0^2 = \frac{1}{2}(\frac{1}{3}8 - \frac{1}{2}4) = \frac{1}{3}$

(4) Compute the following limits or prove they do not exist.

(a) $\lim_{x\to 1} \frac{x}{x-1}\sqrt{1-\frac{2}{x}+\frac{1}{x^2}}$

Solution. Since

$$\sqrt{1-\frac{2}{x}+\frac{1}{x^2}} = \sqrt{\Big(1-\frac{1}{x}\Big)^2} = \sqrt{\Big(\frac{x-1}{x}\Big)^2} = \Big|\frac{x-1}{x}\Big|$$

we find

$$\frac{x}{x-1}\sqrt{1-\frac{2}{x}+\frac{1}{x^2}} = \frac{x}{x-1}\Big|\frac{x-1}{x}\Big| = \begin{cases} -1 & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

This shows that the right-hand limit is 1, and the left-hand limit is -1. Therefore, the original limit does not exist.

(b) $\lim_{x\to 0} \sin(x)\cos(1/x)$

Solution. Since $0 \le |\cos(1/x)| \le 1$, we get $0 \le |\sin(x)\cos(1/x)| \le |\sin(x)|$ and since $\lim_{x\to 0} |\sin(x)| = 0$, the Squeezing Principle implies that $\lim_{x\to 0}\sin(x)\cos(1/x) = 0$

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(5) (a) Find the maximum and the minimum of $f(x) = \dfrac{x^2+1}{x^3+2}$ on $[0, 2]$.

Solution.

$$f'(x) = \frac{2x(x^3+2) - 3x^2(x^2+1)}{(x^3+2)^2} = -\frac{x(x-1)(x^2+x+4)}{(x^3+2)^2}$$

Thus, $f'(x) = 0$ when $x = 0$ or $x = 1$. Evaluating $f(x)$ at these critical points and the endpoints of the interval reveals a minimum value of $f(0) = f(2) = 1/2$ and a maximum of $f(1) = 2/3$.

(b) Find the line tangent to the ellipse $x^2 + 4y^2 = 4$ at the point $(\sqrt{3}, -1/2)$.

Solution. Taking the derivative implicitly, we find $2x + 8yy' = 0$ or $y' = -x/(4y)$. Evaluating the derivative at $(x, y) = (\sqrt{3}, -1/2)$ gives $y' = \sqrt{3}/2$. The equation of the tangent line is thus $y + 1/2 = (\sqrt{3}/2)(x - \sqrt{3})$ or $y = (\sqrt{3}/2)x - 2$.

(6) Use the definition of the derivative to prove that $\dfrac{d}{dx}\sin(x) = \cos(x)$.

Solution. We first observe that

$$\begin{aligned}\lim_{h\to 0}\frac{\cos(h)-1}{h} &= \lim_{h\to 0}\frac{\cos^2(h)-1}{h(\cos(x)+1)} = \lim_{h\to 0} -h\frac{\sin^2(h)}{h^2}\frac{1}{(\cos(x)+1)} \\ &= -0\cdot 1^2 \cdot \frac{1}{2} = 0\end{aligned}$$

Using the addition formula for $\sin(x+h)$ we find

$$\begin{aligned}\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h} \\ &= \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x)\end{aligned}$$

(7) Use the definition of the limit to prove that $\lim_{x\to 3} x^2 = 9$.

Solution. Given any $\epsilon > 0$ let $\delta = \min\{1, \epsilon/7\}$. If $0 < |x-3| < \delta$, then $|x-3| < 1$ and $|x-3| < \epsilon/7$. In particular, $2 < x < 4$, so $|x+3| < 7$. Thus $|x^2 - 9| = |x+3||x-3| < 7|x-3| < 7(\epsilon/7) = \epsilon$.

(8) Use the Mean Value Theorem to prove that if $f'(x) > 0$ on an interval (a, b), then $f(x)$ is strictly increasing on that interval.

Solution. Let x_1, and x_2 be any two points in the interval (a, b) with $x_1 < x_2$. We must show that $f(x_1) < f(x_2)$. The Mean Value Theorem implies that there is a $c \in (x_1, x_2)$ such that $f'(c)(x_2 - x_1) = f(x_2) - f(x_1)$. Since the left hand side is positive, so is the right hand side, and therefore $f(x_1) < f(x_2)$.

(9) (a) Show that if $f(x)$ is a polynomial of degree n, then $f'(x)$ is a polynomial of degree $n - 1$.

Solution. A polynomial a degree n is a function of the form $f(x) = \sum_{k=0}^{n} c_k x^k$ with $c_n \neq 0$. The power rule and linearity of derivatives implies that $f'(x) = \sum_{k=1}^{n} k c_k x^{k-1}$. The highest order term in this sum is $n c_n x^{n-1}$ which means $f'(x)$ has degree $n - 1$.

(b) Use Rolle's Theorem, induction, and part a) to prove that if $f(x)$ is a polynomial of degree n then $f(x)$ has at most n distinct roots.
Solution. Let us prove the assertion using induction on the degree n of $f(x)$. The assertion is true for $n = 1$: In this case, $f(x) = c_1x+c_0$ has exactly one root, $x = -c_0/c_1$. We now assume the assertion is true for polynomials of degree $n-1 \geq 1$, and prove it is true for polynomials of degree n. By part a), if $f(x)$ has degree n, then $f'(x)$ has degree $n-1$. By the induction hypothesis, we know that $f'(x)$ can have at most $n-1$ distinct roots. Rolle's theorem implies that between any two roots of $f(x)$ there is a root of $f'(x)$. So, if $f(x)$ had more than n distinct roots, say $r_1, \ldots, r_m$ with $m > n$, then $f'(x)$ would have more than $n-1$ roots, i.e., at least one root in each interval (r_i, r_{i+1}), $i = 1, \ldots, m-1$. This contradiction implies that $f(x)$ can have at most n distinct roots.

(10) Let $f(x) = \dfrac{2}{5}x^5 - 3x^3 + 7x$.

(a) Determine the intervals on which f is increasing and decreasing.
Solution. The roots of $f'(x) = 2x^4 - 9x^2 + 7$ are $x = \pm 1$ and $x = \pm\sqrt{7/2}$. Moreover, $f'(x) > 0$ and $f(x)$ is increasing for $x < -\sqrt{7/2}$, $-1 < x < 1$, and $x > \sqrt{7/2}$; $f'(x) < 0$ and $f(x)$ is decreasing for $-\sqrt{7/2} < x < -1$, and $1 < x < \sqrt{7/2}$.

(b) Determine the relative maxima and minima of f.
Solution. The FIRST DERIVATIVE TEST implies that $f(x)$ has relative maxima at $x = -\sqrt{7/2}$ and $x = 1$ ($f(-\sqrt{7/2}) = -(7/5)\sqrt{7/2}$, $f(1) = 22/5$), and relative minima at $x = -1$, and $x = \sqrt{7/2}$ ($f(-1) = -22/5$ $f(\sqrt{7/2}) = (7/5)\sqrt{7/2}$).

(c) Determine the intervals on which f is convex and concave.
Solution. The roots of $f''(x) = 8x^3 - 18x$ are $x = \pm 3/2$ and $x = 0$. $f''(x) > 0$ and $f(x)$ is convex for $-3/2 < x < 0$ and $x > 3/2$. $f''(x) < 0$ and $f(x)$ is concave for $x < -3/2$ and $0 < x < 3/2$.

(d) Sketch the graph of f.