MATH 165: HONORS CALCULUS I
ASSIGNMENT 1 SOLUTIONS

Page 19.

Problem 4 Zero has no reciprocal.

Proof. Suppose 0 has a reciprocal, 0^{-1}, so that $0 \cdot 0^{-1} = 1$. This immediately contradicts Theorem I.6, which implies that $0 \cdot 0^{-1} = 0$ ($0 \neq 1$ by Axiom 4). Therefore our assumption that 0 has a reciprocal must be false. $\square$

Problem 5 $-(a+b) = -a - b$

Proof. By definition (I.2), $-a-b$ is the unique number such that $b+(-a-b) = -a$. Adding a to both sides gives $a+b+(-a-b) = a+(-a) = 0$. This equation shows that $-a-b$ is the unique negative of $a+b$, and so $-a-b = -(a+b)$.

$\square$

Problem 8 If $a \neq 0$ and $b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$.

Proof. By definition (I.8), $(ab)^{-1}$ is the unique number such that $(ab)(ab)^{-1} = 1$. But $a^{-1}b^{-1}$ also has this property: $(ab)a^{-1}b^{-1} = aa^{-1}bb^{-1} = 1 \cdot 1 = 1$ (Axioms 1, 2 and 6). Therefore, $(ab)^{-1} = a^{-1}b^{-1}$. $\square$

Problem 9 $-(a/b) = (-a)/b = a/(-b)$ if $b \neq 0$.

Proof. By definition (I.2), $-(a/b)$ is the unique number such that $a/b + (-(a/b)) = 0$. We now show $(-a)/b$ also has this property:

$$\begin{aligned} a/b + (-a)/b &= ab^{-1} + (-a)b^{-1} && \text{(I.9)} \\ &= (a+(-a))b^{-1} && \text{(Axiom 3)} \\ &= 0 \cdot b^{-1} && \text{(I.2)} \\ &= 0 && \text{(I.6)} \end{aligned}$$

Therefore, $(-a)/b = -(a/b)$.
Finally, we prove the same result for $a/(-b)$. First observe that $(-(b^{-1}))(-b) = b^{-1}b = 1$ by I.12 and I.8, so that $(-b)^{-1} = -(b^{-1})$. Then

$$\begin{aligned} a/b + a/(-b) &= ab^{-1} + a(-b)^{-1} && \text{(I.9)} \\ &= a(b^{-1} + (-b)^{-1}) && \text{(Axiom 3)} \\ &= a(b^{-1} - b^{-1}) && \text{(just shown above)} \\ &= a \cdot 0 && \text{(I.2)} \\ &= 0 && \text{(I.6)} \end{aligned}$$

$\square$

Problem 10 $(a/b) - (c/d) = (ad - bc)/(bd)$ if $b \neq 0$ and $d \neq 0$.

Proof. The proof I gave in class was the following: $(ad - bc)/(bd)$, is, by definition (I.8), the unique number that when multiplied by bd gives $ad - bc$. But $(a/b) - (c/d)$ also has this property:

$$\begin{aligned} bd((a/b) - (c/d)) &= bdab^{-1} - bdcd^{-1} && \text{(I.5 and I.9)} \\ &= adbb^{-1} - bcdd^{-1} && \text{(Axiom 1)} \\ &= ad - bc && \text{(I.12)|} \end{aligned}$$

Therefore, $(a/b) - (c/d) = (ad - bc)/(bd)$.
Here's a different proof:

$$\begin{aligned} (a/b) - (c/d) &= a/b + (-c)/d && \text{(Problem 9)} \\ &= (ad + (-c)b)/(bd) && \text{(I.13)} \\ &= (ad - cd)/(bd) && \text{(I.12)|} \end{aligned}$$

□

Page 21.

Problem 2 There is no real number x such that $x^2 + 1 = 0$.

Proof. By I.20, $x^2 > 0$ for any real number x. Then, by I.18, Axiom 4, and I.21, $x^2 + 1 > 0 + 1 = 1 > 0$. In particular, by I.16, $x^2 + 1 \neq 0$ for any real number x. □

Problem 4 If $a > 0$, then $1/a > 0$; if $a < 0$, then $1/a < 0$.

Proof. By definition (I.8) and I.21, $a(1/a) = 1 > 0$. If $a > 0$, then I.24 implies that $1/a > 0$, and if $a < 0$, then I.24 implies that $1/a < 0$. □

Problem 5 If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Proof. By assumption, $a, b \in \mathbb{R}^+$ and $b - a \in \mathbb{R}^+$. Axiom 7 says that $ab \in \mathbb{R}^+$ and Problem 4 implies that $1/b > 0$ and $1/(ab) = (ab)^{-1} > 0$. Since $1/a - 1/b = (b - a)/(ab)$ by Problem 10 on p.19, and since $(b - a)/(ab) = (b - a)(ab)^{-1} > 0$ by I.9 and Axiom 7, we conclude that $1/a - 1/b > 0$. Therefore, $0 < 1/b < 1/a$. □

Problem 9 There is no real number a such that $x \leq a$ for all real x.

Proof. (By contradiction.) Suppose there really is a real number a such that $x \leq a$ for all real numbers x (note that a is fixed and the inequality must hold no matter what x we choose). Let us consider $x = a + 1$. Since $1 > 0$ by I.21, we find that $x = a + 1 > a$ by I.18. This contradicts our assumption that $x \leq a$ (see I.16). Therefore our assumption can never be true and there is no real number a such that $x \leq a$ for all x. □

Problem 10 If x has the property that $0 \leq x < h$ for *every* positive real number h, then $x = 0$.

Proof. Let x be a (fixed) real number that has the above property. By assumption, $x \geq 0$. Suppose $x > 0$, and consider the positive real number $x/2$. We are assuming that x is less than *every* positive real number, including $x/2$. But $x < x/2$ implies that $x < 0$ (use I.18 and I.19). This contradiction implies our assumption that $x > 0$ is false and hence $x = 0$. □