

MATH 165: HONORS CALCULUS I
ASSIGNMENT 9 SOLUTIONS

Problem 1 p.45, #13

a) $b^p - a^p = (b - a) \sum_{k=0}^{p-1} b^{p-k-1} a^k.$

Proof.

$$\begin{aligned} (b - a) \sum_{k=0}^{p-1} b^{p-k-1} a^k &= \sum_{k=0}^{p-1} b b^{p-k-1} a^k - a b^{p-k-1} a^k \\ &= \sum_{k=0}^{p-1} b^{p-k} a^k - b^{p-(k+1)} a^{k+1} \\ &= b^{p-0} a^0 - b^{p-(p-1+1)} a^{p-1+1} \quad [\text{telescoping sum}] \\ &= b^p - a^p \end{aligned}$$

□

b) $n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p$

Proof. Use the formula from a) for the case $p+1$ and plug in $b = n+1$ and $a = n$. This gives

$$(n+1)^{p+1} - n^{p+1} = (n+1 - n) \sum_{k=0}^p (n+1)^{p-k} n^k = \sum_{k=0}^p (n+1)^{p-k} n^k$$

Apply the inequality $n^k < (n+1)^k$ for $k = 1, \dots, p$ on the right hand side to get

$$(n+1)^{p+1} - n^{p+1} < \sum_{k=0}^p (n+1)^{p-k} (n+1)^k = \sum_{k=0}^p (n+1)^p$$

The right hand side consists of $(p+1)$ copies of $(n+1)^p$ so we get

$$(n+1)^{p+1} - n^{p+1} < (p+1)(n+1)^p$$

The other inequality is similar. Apply the inequality $(n+1)^k > n^k$ for $k = 1, \dots, p$ on the right hand side to get

$$\begin{aligned} (n+1)^{p+1} - n^{p+1} &= \sum_{k=0}^p (n+1)^{p-k} n^k \\ &> \sum_{k=0}^p n^{p-k} n^k = \sum_{k=0}^p n^p \\ &= (p+1)n^p \end{aligned}$$

□

$$\text{c) } \sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^n k^p \quad (*)$$

Proof. Use induction on n . The formula is true for $n = 1$:

$$0 = \sum_{k=1}^0 k^p < \frac{1^{p+1}}{p+1} < \sum_{k=1}^1 k^p$$

We assume the inequalities $(*)$ are true for n , and prove they hold for $n+1$. The left hand side of $(*)$ for $n+1$ looks like:

$$\begin{aligned} \sum_{k=1}^n k^p &= \left(\sum_{k=1}^{n-1} k^p \right) + n^p \\ &< \frac{n^{p+1}}{p+1} + n^p \quad [\text{induction hypothesis}] \\ &< \frac{(n+1)^{p+1}}{p+1} \quad [\text{by part b)}] \end{aligned}$$

so the left inequality is true for $n+1$. The right hand side of $(*)$ for $n+1$ is:

$$\begin{aligned} \sum_{k=1}^{n+1} k^p &= \left(\sum_{k=1}^n k^p \right) + (n+1)^p \\ &> \frac{n^{p+1}}{p+1} + (n+1)^p \quad [\text{induction hypothesis}] \\ &> \frac{(n+1)^{p+1}}{p+1} \quad [\text{by part b)}] \end{aligned}$$

□

Problem 2 Let A and B be non-empty sets of real numbers bounded above and below, respectively.

a) If $k < \sup A$ then there is a number $a \in A$ such that $k < a$.

Proof. Suppose statement a) is false. Then there is some $k < \sup A$ such that $k \geq a$ for all $a \in A$. Therefore, k is an upper bound for A . Since $\sup A$ is the *least* upper

bound for A , it must be less than k : $\sup A \leq k$. But this contradicts $k < \sup A$. Therefore statement a) must be true. \square

b) If $h > \inf B$ then there is a number $b \in B$ such that $b < h$.

Proof. Suppose statement b) is false. Then there is some $h > \inf B$ such that $b \geq h$ for all $b \in B$. Therefore, h is a lower bound for B . Since $\inf B$ is the *greatest* lower bound for B , it must be greater than h : $\inf B \geq h$. But this contradicts $h > \inf B$. Therefore, statement b) must be true. \square

Problem 3 Let A and B be non-empty sets of real numbers with A contained in B .

a) If A and B are bounded above, then $\sup A \leq \sup B$.

Proof. Let $k = \sup B$ and suppose $k < \sup A$. By 2a), there is a number $a \in A$ such that $k < a$. Since $k = \sup B \geq b$ for all $b \in B$, the number a cannot be in B . But this contradicts the assumption that B contains A . Since assuming $k < \sup A$ leads to a contradiction, it must be that $\sup A \leq k$, i.e., $\sup A \leq \sup B$. \square

b) If A and B are bounded below, then $\inf B \leq \inf A$.

Proof. Let $h = \inf B$ and suppose $h > \inf A$. By 2b), there is a number $a \in A$ such that $a < h$. Since $h = \inf B \leq b$ for all $b \in B$, the number a cannot be in B . But this contradicts the assumption that B contains A . Since assuming $h > \inf A$ leads to a contradiction, it must be that $h \leq \inf A$, i.e., $\inf B \leq \inf A$. \square

Problem 4 For $0 \leq x \leq 1$ define

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

a) If s and t are step functions satisfying $s(x) \leq f(x) \leq t(x)$ for all $0 \leq x \leq 1$, then $s(x) \leq 0$ and $t(x) \geq 1$, except possibly at partition points.

Proof. Let $P = \{x_0, x_1, \dots, x_n\}$ be a common refinement of partitions for s and t so that s and t are constant on the open subintervals of P : $s(x) = s_k$ and $t(x) = t_k$ for $x_{k-1} < x < x_k$, $k = 1, \dots, n$. By assumption, $s_k \leq f(x) \leq t_k$ for $x_{k-1} < x < x_k$. Let q_k be a rational number and r_k an irrational number in the open interval (x_{k-1}, x_k) , for $k = 1, \dots, n$. Then $s_k \leq f(r_k) = 0$ and $1 = f(q_k) \leq t_k$ for $k = 1, \dots, n$. Therefore, $s(x) \leq 0$ and $t(x) \geq 1$, except possibly at partition points. \square

b) The lower integral of f , $\underline{I}(f) = 0$, and the upper integral of f , $\bar{I}(f) = 1$. Hence f cannot be integrable by Theorem 1.9.

Proof. Let s and t be step functions as in part a). Then

$$\int_0^1 s(x) dx = \sum_{k=1}^n s_k(x_k - x_{k-1}) \leq \sum_{k=1}^n 0(x_k - x_{k-1}) = 0$$

and

$$\int_0^1 t(x) dx = \sum_{k=1}^n t_k(x_k - x_{k-1}) \geq \sum_{k=1}^n 1(x_k - x_{k-1}) = 1.$$

We conclude that every number in the set

$$S = \left\{ \int_0^1 s(x) dx \mid s \leq f \right\}$$

is ≤ 0 and every number in the set

$$T = \left\{ \int_0^1 t(x) dx \mid f \leq t \right\}$$

is ≥ 1 . Therefore $\underline{I}(f) = \sup S \leq 0$ and $\bar{I}(f) = \inf T \geq 1$. Notice also that the constant step functions $s(x) = 0$ and $t(x) = 1$ satisfy $s(x) \leq f(x) \leq t(x)$ for all $0 \leq x \leq 1$, so $0 \in S$ and $1 \in T$. Therefore $\underline{I}(f) = \sup S = 0$ and $\bar{I}(f) = \inf T = 1$. \square