MATH 165: HONORS CALCULUS I
ASSIGNMENT 10 SOLUTIONS

Problem 1 Let A and B be subsets of real numbers each of which has an infimum. Let $C = \{a+b \,|\, a \in A,\ b \in B\}$. Prove that C has an infimum and $\inf C = \inf A + \inf B$. (The proof of the corresponding theorem for supremum, Theorem I.33 (a), is given on p.27.)

Proof. Any $c \in C$ has the form $c = a + b$ where $a \in A$ and $b \in B$. Since $a \geq \inf A$ and $b \geq \inf B$, we get $c = a + b \geq \inf A + \inf B$, which shows that $\inf A + \inf B$ is a lower bound for C. By Theorem I.27, C has an infimum, $\inf C$, which is greater than or equal to any lower bound of C, so $\inf C \geq \inf A + \inf B$. If $\inf C$ is strictly greater than $\inf A + \inf B$ then $\inf C = \inf A + \inf B + h$ for some positive number $h > 0$. By Assignment 9, 2b) (see also Theorem I.32), there is an $x \in A$ and a $y \in B$ such that $x < \inf A + h/2$ and $y < \inf B + h/2$. Since $x + y \in C$,

$$\inf C \leq x + y < (\inf A + h/2) + (\inf B + h/2) = \inf A + \inf B + h = \inf C$$

But $\inf C < \inf C$ is a contradiction! Therefore, $\inf C = \inf A + \inf B$. $\square$

Problem 2 Define

$$f(x) = \begin{cases} 0 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

Prove that $f(x)$ is integrable on $[0, 1]$.

Proof. Fix $n \in \mathbb{N}$ and define a step function s_n by

$$s_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/n \\ f(x) & \text{if } 1/n < x \leq 1 \end{cases}$$

A partition for s_n is given by the points $x_0 = 0$, $x_1 = 1/n$, $x_2 = 1/(n-1)$, $\ldots$, $x_{n-1} = 1/2$, $x_n = 1$. Define a second step function by $t(x) = 1$ for $0 \leq x \leq 1$. Then

$$s_n(x) \leq f(x) \leq t(x) \text{ for all } 0 \leq x \leq 1$$

The integrals of these steps functions are easy to calculate:

$$\begin{aligned} \int_0^1 s_n &= \sum_{k=1}^{n} s_k(x_k - x_{k-1}) = 0(x_1 - x_0) + \sum_{k=2}^{n} 1(x_k - x_{k-1}) \\ &= x_n - x_1 = 1 - \frac{1}{n} \\ \int_0^1 t &= 1 \end{aligned}$$

Since $\int_0^1 s \leq \underline{I}(f) \leq \overline{I}(f) \leq \int_0^1 t$ for any step functions satisfying $s(x) \leq f(x) \leq t(x)$ on $[0, 1]$, we see that

$$1 - \frac{1}{n} \leq \underline{I}(f) \leq \overline{I}(f) \leq 1$$

and these inequalities must hold for any $n \in \mathbb{N}$. Since $I = 1$ is the only number that satisfies

$$1 - \frac{1}{n} \leq I \leq 1 \quad \text{for all } n \in \mathbb{N}$$

we conclude that $\underline{I}(f) = \overline{I}(f) = 1$. Therefore f is integrable and $\displaystyle\int_0^1 f = 1$. $\square$

Problem 3 Use Theorem 1.14 to find an approximation to the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$ so that the error is $< .05$.

Solution. The function $f(x) = \dfrac{1}{x}$ is decreasing on $[1, 2]$. The partition points for n subintervals of equal length are given by $x_k = 1 + k\dfrac{1}{n}$, for $k = 1, \ldots, n$. Let

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1 + k\frac{1}{n}} = \sum_{k=1}^{n} \frac{1}{n+k}$$

By Theorem 1.14,

$$\sigma_n \leq \int_1^2 \frac{1}{x}\,dx \leq \sigma_n + \varepsilon_n$$

where

$$\varepsilon_n = \frac{(2-1)(f(1) - f(2))}{n} = \frac{1}{2n}$$

If we take our approximation to be the midpoint of the interval $[\sigma_n, \sigma_n + \varepsilon_n]$, namely $\alpha_n = \sigma_n + \varepsilon_n/2$, then the true value of the integral will lie with $\pm\varepsilon_n/2$ of α_n. We would like this error to be less than 0.05:

$$\frac{1}{2}\varepsilon_n < 0.05 \Longleftrightarrow \frac{1}{4n} < 0.05 \Longleftrightarrow n > \frac{1}{4 \cdot 0.05} = 5$$

Therefore, the desired approximation is

$$\begin{aligned}
\alpha_6 &= \sigma_6 + \varepsilon_6/2 \\
&= \left(\frac{1}{6+1} + \frac{1}{6+2} + \frac{1}{6+3} + \frac{1}{6+4} + \frac{1}{6+5} + \frac{1}{6+6}\right) + \frac{1}{2}\left(\frac{1}{2 \cdot 6}\right) \\
&= 0.694877
\end{aligned}$$

The actual value of this integral is $\log(2) = 0.693147$ and the error of our approximation is $0.694877 - 0.693147 = 0.00173 < 0.05$.