MATH 165: HONORS CALCULUS I
ASSIGNMENT 15 SOLUTION

Graph $F(x) = \int_0^x [t]\, dt$ on $[-4, 4]$.

Solution. The graph of $[t]$ for $-4 \leq t \leq 4$:

First, let us assume $x \geq 0$. Using additivity of intervals and letting $n = [x]$:

$$\begin{aligned} F(x) &= \int_0^x [t]\, dt = \int_0^1 0\, dt + \int_1^2 1\, dt + \int_2^3 2\, dt + \cdots + \int_{n-1}^n (n-1)\, dt + \int_n^x n\, dt \\ &= 0 + 1 + 2 + \cdots + (n-1) + n(x-n) \\ &= (n-1)n/2 + nx - n^2 = n(x - (n+1)/2) \end{aligned}$$

Therefore, $F(x) = [x](x - ([x]+1)/2)$. Of course, $[t]$ is a step function and this result could also be obtained using the partition $\{0, 1, \ldots, n, x\}$ and the usual formulas for step functions.
If $x < 0$:

$$\begin{aligned} \int_0^x [t]\, dt &= \int_0^{-1} (-1)\, dt + \int_{-1}^{-2} (-2)\, dt + \int_{-2}^{-3} (-3)\, dt + \cdots + \int_{-(n-2)}^{-(n-1)} (-(n-1))\, dt + \int_{-(n-1)}^{x} (-n)\, dt \\ &= (-1)(-1) + (-2)(-1) + (-3)(-1) \cdots + (-(n-1))(-1) + (-n)(x + (n-1)) \\ &= (n-1)n/2 - nx - n(n-1) = n(-x - (n-1)/2) \end{aligned}$$

In this case, $n = -[x]$, so $F(x) = -[x](-x - (-[x]-1)/2 = [x](x - ([x]+1)/2)$ which is the same expression as for the case $x > 0$. Here is the graph of $F(x) = [x](x - ([x]+1)/2)$ for $-4 \leq x \leq 4$: