MATH 165: HONORS CALCULUS I
ASSIGNMENT 16 SOLUTIONS

Problem 1 Show that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Solution. Given $\epsilon > 0$ we must find a $\delta > 0$ (that may depend on ϵ) such that if $0 < x - 0 < \delta$, then $|\sqrt{x} - 0| < \epsilon$. Since $0 < x < \delta$ implies that $0 < \sqrt{x} < \sqrt{\delta}$ and we want to force $\sqrt{x} < \epsilon$, we should simply choose $\delta = \epsilon^2$. Then $0 < x < \delta$ implies $0 < \sqrt{x} < \sqrt{\epsilon^2} = \epsilon$.

Problem 2 Show that $\lim_{x\to 0^+} x^{1/n} = 0$ for $n \in$.

Solution. This is similar to Problem 1. Given $\epsilon > 0$ we must find a $\delta > 0$ such that if $0 < x - 0 < \delta$, then $|x^{1/n} - 0| < \epsilon$. Since $0 < x < \delta$ implies that $0 < x^{1/n} < \delta^{1/n}$ and we want to force $x^{1/n} < \epsilon$, we should simply choose $\delta = \epsilon^n$. Then $0 < x < \delta$ implies $|x|^{1/n} < (\epsilon^n)^{1/n} = \epsilon$.

Problem 3 Show that $\lim_{x\to 0^-} x^{1/n} = 0$ for n an odd positive integer.

Solution. The proof is similar to the one in Problem 2, but we have to be careful of negative numbers. For example, n has to be odd to be able to define $x^{1/n}$ for $x < 0$. Given $\epsilon > 0$ we must find a $\delta > 0$ such that if $-\delta < x < 0$, then $|x^{1/n} - 0| < \epsilon$. Since $-\delta < x < 0$ implies $0 < |x| < \delta$ and $|x|^{1/n} < \delta^{1/n}$, and since we want to force $|x|^{1/n} < \epsilon$, we should simply choose $\delta = \epsilon^n$. Then $-\delta < x < 0$ implies $|x| < \delta = \epsilon^n$ and hence $|x|^{1/n} < \epsilon$.

Problem 4 Let $f(x) = \frac{|x|}{x}$ for $x \neq 0$. Use one-sided limits to show that $\lim_{x\to 0} f(x)$ does not exist.

Solution. If $x < 0$, then $f(x) = \frac{|x|}{x} = -1$. Since we are allowed to assume $x < 0$ in the left-hand limit $\lim_{x\to 0^-} f(x)$, it follows that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -1 = -1$. Similarly, if $x > 0$, then $f(x) = \frac{|x|}{x} = 1$. So $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$. Since the left-hand limit does not equal the right hand limit, the limit itself, $\lim_{x\to 0} f(x)$, cannot exist.

Problem 5 Let $f(x) = x^2$ and let $p \in$.

a) Show that if $|x - p| < 1$ then $|x| < |p| + 1$.

Solution. If $|x - p| < 1$, then $||x| - |p|| \leq |x - p| < 1$. This implies $|p| - 1 < |x| < |p| + 1$.

b) Show that if $|x - p| < 1$ then $|x^2 - p^2| < (2|p| + 1)|x - p|$.

Solution. If $|x - p| < 1$ then $|x + p| \leq |x| + |p| < (|p| + 1) + |p| = 2|p| + 1$ (by the triangle inequality and part a)). Therefore, if $|x - p| < 1$, then $|x^2 - p^2| = |x + p||x - p| < (2|p| + 1)|x - p|$.

c) Show that $\lim_{x\to p} x^2 = p^2$.

Solution. We must show that given any $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - p| < \delta$ then $|x^2 - p^2| < \epsilon$. From part b) we know that if $|x - p| < 1$ then $|x^2 - p^2| < (2|p| + 1)|x - p|$. We want to force $|x^2 - p^2| < \epsilon$ and we can do this by making $(2|p| + 1)|x - p| < \epsilon$, or $|x - p| < \epsilon/(2|p| + 1)$. Since we need two constraints to make this work, namely $|x - p| < 1$ and $|x - p| < \epsilon/(2|p| + 1)$, we choose δ to be the smaller: $\delta = \min(1, \epsilon/(2|p| + 1))$. Therefore, if $|x - p| < \delta$, then $|x - p| < 1$ *and* $|x - p| < \epsilon/(2|p| + 1)$. Together, these imply that $|x^2 - p^2| < (2|p| + 1)|x - p| < (2|p| + 1) \cdot \epsilon/(2|p| + 1) = \epsilon$.