MATH 165: HONORS CALCULUS I
ASSIGNMENT 20 SOLUTIONS

Problem 6 Suppose f and g are functions such that $g(f(x)) = x$ for all x in the domain of f. Prove that f is one-to-one.

Proof. We must show that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. Suppose this statement were not true. Then there would be numbers $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. But then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$, a contradiction. Therefore, the statement must be true. □

Problem 7 Let $f(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$ for $x > 0$.

a) Prove that f is strictly increasing and $f^{-1}(0) = 1$.

Proof. Let $0 < x_1 < x_2$. We must show that $f(x_1) < f(x_2)$. But this follows from:

$$f(x_2) = \int_1^{x_2} \frac{1}{t}\,dt = \int_1^{x_1} \frac{1}{t}\,dt + \int_{x_1}^{x_2} \frac{1}{t}\,dt = f(x_1) + \int_{x_1}^{x_2} \frac{1}{t}\,dt$$

Since $\frac{1}{t} \geq \frac{1}{x_2}$ on the interval $[x_1, x_2]$, $\displaystyle\int_{x_1}^{x_2} \frac{1}{t}\,dt \geq \frac{1}{x_2}(x_2 - x_1) > 0$, and so $f(x_2) > f(x_1)$. Finally, $f^{-1}(0) = 1$ because $f(1) = \displaystyle\int_1^1 \frac{1}{t}\,dt = 0$. □

b) Prove that $f(ab) = f(a) + f(b)$.

Proof. $f(ab) = \displaystyle\int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt = f(a) + \int_a^{ab} \frac{1}{t}\,dt$. By Theorem 1.19, $\displaystyle\int_a^{ab} \frac{1}{t}\,dt = a\int_1^b \frac{1}{at}\,dt = \int_1^b \frac{1}{t}\,dt = f(b)$. □

c) Prove that $f^{-1}(a + b) = f^{-1}(a)f^{-1}(b)$.

Proof. Let $c = f^{-1}(a)$ and $d = f^{-1}(b)$. Then $f(c) = a$ and $f(d) = b$. Using part b) we find that $f^{-1}(a + b) = f^{-1}(f(c) + f(d)) = f^{-1}(f(cd)) = cd = f^{-1}(a)f^{-1}(b)$. □

Problem 8 Let f be strictly decreasing on $[a, b]$. Prove that the inverse f^{-1} is strictly decreasing on $[f(b), f(a)]$.

Proof. Let $y_1 < y_2$ in $[f(b), f(a)]$. We must show that $f^{-1}(y_1) > f^{-1}(y_2)$. Let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$ so $f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is strictly decreasing, $x_1 > x_2$ (if $x_1 \leq x_2$ we would have $y_1 = f(x_1) \geq f(x_2) = y_2$, a contradiction). Therefore, $f^{-1}(y_1) = x_1 > x_2 = f^{-1}(y_2)$. □

Problem 9 Let $f(x) = x^4 - 8x^2 + 8$. Determine the intervals on which $f(x)$ is strictly increasing and strictly decreasing and find the inverse of $f(x)$ on each of those intervals. Plot $f(x)$ and each of the inverses as functions of x.

Solution. It is clear from the graph that $f(x) = (x^2 - 4)^2 - 8$ is strictly decreasing on the intervals $(-\infty, -2]$ and $[0, 2]$, and strictly increasing on the intervals $[-2, 0]$ and $[2, \infty)$. To prove this analytically: if $x_1 < x_2 \leq -2$, then $x_1^2 > x_2^2 \geq 4$ so $x_1^2 - 4 > x_2^2 - 4 \geq 0$. Therefore, $(x_1^2 - 4)^2 > (x_2^2 - 4)^2$

and $f(x_1) > f(x_2)$. If $-2 \le x_1 < x_2 \le 0$, then $4 \ge x_1^2 > x_2^2$ so $0 \ge x_1^2 - 4 > x_2^2 - 4$. Hence $(x_1^2-4)^2 < (x_2^2-4)^2$ and $f(x_1) < f(x_2)$. If $0 \le x_1 < x_2 \le 2$, then $x_1^2 < x_2^2 \le 4$ so $x_1^2-4 < x_2^2-4 \le 0$. It follows that $(x_1^2 - 4)^2 > (x_2^2 - 4)^2$ and $f(x_1) > f(x_2)$. Finally, if $2 \le x_1 < x_2$, then $4 \le x_1^2 < x_2^2$ so $0 \le x_1^2 - 4 < x_2^2 - 4$. Thus, $(x_1^2 - 4)^2 < (x_2^2 - 4)^2$ and $f(x_1) < f(x_2)$.
To find the inverses of f on these intervals we solve the equation $y = (x^2 - 4)^2 - 8$ for x to get $x = \pm\sqrt{4 \pm \sqrt{y + 8}}$.

(1) If $x < -2$, then x is negative and $|x| > 2$, forcing the outside sign to be negative and the inside sign to be positive (otherwise the square root would be less than 2): $x = -\sqrt{4 + \sqrt{y + 8}}$. As a function of x the inverse is: $f^{-1}(x) = -\sqrt{4 + \sqrt{x + 8}}$ for $x \ge -8$ (the domain of the inverse is the range of the original function f on the interval $(-\infty, -2)$).

(2) If $-2 \le x \le 0$, then x is still negative, but we need the square root to be less than 2, so $x = -\sqrt{4 - \sqrt{y + 8}}$. As a function of x the inverse is: $f^{-1}(x) = -\sqrt{4 - \sqrt{x + 8}}$ for $-8 \le x \le 8$.

(3) If If $0 \le x \le 2$, then x is positive so the outside sign is positive, but we still need the square root to be less than 2 so $x = +\sqrt{4 - \sqrt{y + 8}}$. As a function of x the inverse is: $f^{-1}(x) = +\sqrt{4 - \sqrt{x + 8}}$ for $-8 \le x \le 8$

(4) Finally, if $x \ge 2$, the both signs should be positive so the square root will be greater than 2, $x = +\sqrt{4 + \sqrt{y + 8}}$ As a function of x the inverse is: $f^{-1}(x) = +\sqrt{4 + \sqrt{x + 8}}$ for $x \ge -8$

Problem 10 Determine the largest interval containing 0 on which the functions $\sin(x)$ and $\tan(x)$ have inverses and plot the inverses on those intervals. Explain why there is no interval containing 0 (i.e., no interval of the form $[a, b]$ where $a < 0$ and $b > 0$) on which $\cos(x)$ has an inverse. Then find an interval on which $\cos(x)$ *does* have an inverse and plot the inverse there.

Solution. $\sin(x)$ and $\tan(x)$ are strictly increasing on $[-\pi/2, \pi/2]$, but on no larger interval comtaining 0. The range of $\sin(x)$ on this intevral is $[-1, 1]$ which becomes the domain of its inverse function.

The range of $\tan(x)$ on this interval is $(-\infty, \infty)$, which means its inverse is defined for all real numbers.

$\cos(x)$ is increasing on $[-\pi, 0]$ and decreasing on $[0, \pi]$, so is cannot be monotonic on any interval containing 0 as an interior point. The natural interval to restrict to is $[0, \pi]$. The range of $\cos(x)$ on this interval is $[-1, 1]$, so becomes the domain of its inverse function.