COVERING REGIONS WITH A HEXAGON

MATH 165: CALCULUS I

Any region of diameter 1 is contained in a region of constant width 1 (any two parallel lines tangent to the region must be a distance of 1 apart). In this note I'll sketch the proof of the following fact:

> *Any region R of constant width 1 can be covered by a regular hexagon whose opposite sides are a distance of 1 apart.*

Take a 60 degree angle, oriented in any direction, and move it towards the region R until both sides of the angle are tangent to R. Next bring a line parallel to AB up towards the region until it is tangent to R, and bring another line parallel to AC towards R until it is tangent to R. We get a figure like the following:

We now have R inscribed in a parallelogram whose opposite sides must be a distance of 1 apart since R has constant width 1. This means that the parallelogram is actually a rhombus—the sides must have equal length. Since the triangles ABC and BDC are isosceles, and the angles at A and D are 60 degrees, it follows that the angles at B and C must be 120 degrees.

Now take two parallel lines that are perpendicular to the line AD and move them towards R until they are both tangent to R. These lines must also be a distance of 1 apart since R has constant width. We now have R inscribed in (a possibly non-regular) hexagon, $EBFGCH$, whose opposite sides are a distance of 1 apart.

Since the line AD bisects the 60 degree angle at A and is perpendicular to EH, the angles AEH and HEA must also be 60 degrees. Therefore, the angles CHE and BEH are 120 degrees. A similar argument implies that the angles BFG and CGF are 120 degrees. Thus, the interior angles of the hexagon are all 120 degrees. The hexagon need not be regular, since the sides do not necessarily have the same length. However, since the interior angles are all the same, if EH and FG have the same length, then all the sides have the same length and $EBFGCH$ is a regular hexagon.

Recall that at the start of the construction, we chose an arbitrary orientation for the angle A. We can repeat the construction for any starting orientation of A and the lengths of EH and FG would change accordingly. Suppose $EH < FG$, as in the above diagram. Then by rotating the original orientation of the angle around 180 degrees we would end up with a diagram like the one below where $EH > FG$. Since the lengths EH and FG change continuously, there must be a starting orientation for the angle A for which the above construction gives $EH = FG$. The resulting hexagon $EBFGCH$ is then the desired regular hexagon whose opposite sides are a distance of 1 apart that contains R.

Date: Fal, 1999.