EXAMPLES OF A LIMIT NOT EXISTING

MATH 165: HONORS CALCULUS I

The definition of $\lim_{x \to p} f(x) = A$ is:

> For all $\epsilon > 0$ there exists a $\delta > 0$ such that if x is any number satisfying $0 < |x - p| < \delta$ then $|f(x) - A| < \epsilon$.

The easiest way to show a limit does not exist is to calculate the one-sided limits, if they exist, and show they are not equal. For example, if $f(x) = [x]$, then $\lim_{x \to 2^-} f(x) = 1$ and $\lim_{x \to 2^+} f(x) = 2$, so $\lim_{x \to 2} f(x)$ does not exist.

It is possible that the one-sided limits do not exist either and the question arises, what exactly do you have to prove to show a limit does not exist. The negation of the definition given above is somewhat complicated and leads to the following:

> For any A, there exists an $\epsilon > 0$ such that for all $\delta > 0$, there is an x satisfying $0 < |x - p| < \delta$ and $|f(x) - A| \geq \epsilon$.

The idea is that for any proposed "limit" A, we can prove that there are points x close to p ($|x - p| < \delta$), for which $f(x)$ is far away from A ($|f(x) - A| \geq \epsilon$).

Example 1: Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

For any number p, the limit $\lim_{x \to p} f(x)$ does not exist.

Proof: Let A be any number. Case 1): Assume $A \neq 0$. Then set $\epsilon = |A|/2 > 0$. No matter what the value of δ is, there is a rational number q satisfying $|q - p| < \delta$ (any interval contains a rational number) and hence $|f(q) - A| = |0 - A| = |A| > |A|/2 = \epsilon$. Case 2) Assume $A = 0$. Then set $\epsilon = 1/2$. No matter what the value of δ is, there is an irrational number r satisfying $|r - p| < \delta$ (any interval contains a irrational number) and hence $|f(r) - A| = |1 - 0| = 1 > 1/2 = \epsilon$.

Example 2: $\lim_{x \to 0} \sin(1/x)$ does not exist.

Proof: Regardless of what A is given, we can take $\epsilon = 1$. Case 1): Assume $A \geq 0$. For any $\delta > 0$ there is a positive integer n such that $x = 1/(3\pi/2 + 2n\pi) < \delta$. Then $\sin(1/x) = \sin(3\pi/2) = -1$ so $|\sin(1/x) - A| \geq 1 = \epsilon$. Case 2): Assume $A < 0$. For any $\delta > 0$, there is a positive integer n such that $x = 1/(\pi/2 + 2n\pi) < \delta$. Then $\sin(1/x) = \sin(\pi/2) = 1$, so $|\sin(1/x) - A| > 1 = \epsilon$.

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