MATH 165 FINAL 12/16/02

NAME:

1) (30 pts.) Write a proof of the following theorem (state the theorem here):

2) (20 points) Give the ϵ- δ definition of continuity of $f : \mathbb{R} \to \mathbb{R}$ at a point $c \in \mathbb{R}$.

3) (20 pts.) Define what is meant by denseness of the rationals (as well as irrationals) on the real line. Is denseness a consequence of the purely algebraic axioms for $\mathbb{R}$? Explain.

4) (20 pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 0$ if $x \leq 0$ and $f(x) = \sqrt{x}$ if $x > 0$. Using the ϵ- δ definition of continuity, show that f is continuous at $a = 0$.

5) (20 pts.) State the principle of mathematical induction and prove that for every $n \in \mathbb{N}$ one has

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}.$$

6) (20 pts) a) Using the formula $\sin(a+b) - \sin(a-b) = 2 \sin a \cos b$ show that

$$\int_0^{2\pi} \sin nx \ \cos mx \ \ dx = 0,$$

where $m, \ n \in \mathbb{Z}$.

b) Compute

$$\int_0^{2\pi} (\sin 3x + \cos 3x)^2 dx.$$

7) (20 pts.) Recall that every rational number x can be written uniquely as $x = \frac{m}{n}$, where m and n are integers, $n > 0$, and the fraction is in its lowest terms. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if x is irrational and $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$ is a rational number written as above.

a) Show that f is continuous at every irrational number. *Hint* : Show first that for any $M > 0$, any bounded interval contains only finitely many rationals with denominator $n \leq M$. Next, think about what it means to say that "δ cannot be found" and use denseness of the rationals...

b) Show that f is discontinuous at every rational number. *Hint* : Use denseness of the irrationals.