amsppt

Improper Integrals

Assume that $f:[a,\infty)\longrightarrow R$ is a function such that for any $b\in(a,\infty)$, f is integrable on $[a,b]$. If

$$\lim_{b\to\infty}\int_a^b f(x)dx.$$

exists and is finite, then we say that the <u>improper integral</u> of f on (a,∞) converges and we write

$$\int_a^\infty f(x)dx=\lim_{b\to\infty}\int_a^b f(x)dx.$$

<u>Example.</u> Let $p>0$. If $p\neq 1$ then

$$I(b)=\int_1^b 1x^p dx=x^{-p+1}-p+1|_1^b=11-p[b^{1-p}-1]$$

If $p>1$ then $I(b)\longrightarrow 1p-1$ as $b\longrightarrow\infty$ and therefore

$$\int_1^\infty 1x^p dx=1p-1,\ \ p>1.$$

If $0<p<1$, then $I(b)\longrightarrow\infty$ and the improper integral $\int_1^\infty 1x^p dx$ diverges. If $p=1$ then $I(b)=lnx|_1^b=lnb\longrightarrow\infty$ as $b\longrightarrow\infty$.
Therefore we have the

<u>Theorem 1</u>

$$\int_1^\infty 1x^p dx=\{\ 1p-1if$$

p ¿ 1

$$I(b)=\int_0^b e^{-ax}dx=-1ae^{-ax}|_0^b=1a(1-e^{-ab})\longrightarrow 1a\ \ as\ b\longrightarrow\infty.$$

Therefore, $\int_0^\infty e^{-ax}dx$ converges to $1a$ if $a>0$. It diverges if $a<0$.
Similarly, we define

$$\int_{-\infty}^b f(x)dx=\lim_{a\to-\infty}\int_a^b f(x)dx$$

if f is integrable on $[a,b]$ for any $a\leq b$ and the above limits exist and is finite.
Also, we define

$$\int_{-\infty}^\infty f(x)dx=\lim_{b\to\infty}\int_c^b f(x)dx+\lim_{a\to-\infty}\int_a^c f(x)dx$$

if f is integrable on any interval $[a,b]$ and both limits above exist and are finite.

If a function f is unbounded and on (a,b] it is integrable on $[a+\epsilon, b]$ for any $\epsilon > 0$ and $\lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} f(x)dx$ exists and is finite, then we say that the improper integral of f over $(a, b]$ converges and we write

$$\int_{a}^{b} f(x)dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} f(x)dx.$$

Example. Let $p > 0$. We have

$$I(\epsilon) = \int_{\epsilon}^{1} 1x^p dx = \{\ 11 - p[a - \epsilon^{1-p}] if\ p \neq 1$$

If $0 < \epsilon < 1$ then $I(\epsilon) \longrightarrow 11 - p$ as $\epsilon \longrightarrow 0$ and if $1 \leq p$ then $I(\epsilon) \longrightarrow \infty$ as $\epsilon \longrightarrow 0$.

Therefore, we have

Theorem 2.

$$\int_{0}^{1} 1x^p dx = \{\ 11 - p if$$

0 ¡ p ¡ 1

If a function f is integrable on $[a, b-\epsilon]$ for any $\epsilon > 0$ and unbounded on $[a, b)$, then we define

$$\int_{b}^{a} f(x)dx = \lim_{\epsilon \longrightarrow 0} \int_{a}^{b-\epsilon} f(x)dx$$

if the above limit exists and is finite.

Finally, we define

$$\int_{a}^{b} f(x)dx = \lim_{\epsilon \longrightarrow 0} \int_{a+\epsilon}^{c} f(x)dx + \lim_{\epsilon \longrightarrow 0} \int_{c}^{b-\epsilon} f(x)dx$$

if f is integrable on any interval $[a+\epsilon,\ b-\epsilon]$ but unbounded on $(a, c]$ and $[c, b)$, $c\epsilon(a, b)$, and if both limits above exist and are finite.

Example. The integral

$$\int_{-1}^{1} 1(1 - x^2)^1 4$$

is improper since

$$f(x) = 1(1 - x^2)^1 4$$

is unbounded on both intervals $[0, 1)$ and $(-1, 0]$. We will show later that it converges.

Convergence Tests for Improper Integrals

Here we will discuss only infinite integrals of the form

$$\int_a^\infty f(x)dx$$

where the function f is integrable over any interval $[a,b], b \geq a$. The tests for the integrals of the form $\int_{-\infty}^a f(x)dx$ are reduced to the above case by the substitution $t = -x$. Also, the substitution $t = 1x - a$ or $t = 1x - b$ reduces the improper integrals $\int_a^b f(x)dx$ to the above case.

Theorem 1 (Cauchy criterion). Suppose that the function f is integrable over any interval $[0,b), b \leq a$. Then the improper integral $\int_a^\infty f(x)dx$ converges if and only if for every $\epsilon > 0$ there exists $N = N(\epsilon)$ such that

$$d \geq c \geq N \Longrightarrow |\int_c^d f(x)dx| < \epsilon.$$

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Remark. In other words, the Cauchy criterion says that $\int_a^\infty f(x)dx$ converges if and only if

$$\int_c^d f(x)dx \longrightarrow 0 \ as\ c \longrightarrow \infty.$$

Example. The infinite integral $\int_1^\infty dxx^2dx$ converges since

$$|\int_c^d dxx^2| = 1d - 1c \leq 2c \longrightarrow 0 \ as \ \ c \longrightarrow \infty.$$

Example. The integral $\int_1^\infty dxx$ does not converge since for $c \geq 1$ we have

$$|\int_c^{c^2} dxx| = ln\ c^2 - ln\ c = ln\ c \longrightarrow \infty\ as\ c \longrightarrow \infty.$$

Proof. $Suppose first that the integral converges, i.e. I(b) = \int_a^b f(x)dx \longrightarrow I \ \ as\ b \longrightarrow \infty.$ Then by the definition of the limit for any $\epsilon > 0$ there exists $N = N(\epsilon) \geq a$ such that

$$b \geq N \Rightarrow |I(b) - I| < \epsilon 2$$

Therefore, if $d \geq c \geq N$ then

$$|\int_c^d f(x)dx| = |I(d) - I(c)| \leq |I(d) - I| + |I(c) - I| < \epsilon 2 + \epsilon 2 = \epsilon$$

Conversely, we assume that (*) holds. The sequence of numbers $I(k)$ is a Cauchy sequence. Such sequences always converge. This is a fact following from the construction of the real numbers. Therefore, there exists a real number I such that $I(k) \longrightarrow I$ as $k \to \infty$. Thus for any $b \geq a$ we have

$$I(b) = [I(b) - I(k)] + I(k) \longrightarrow I \; if \;\; k \geq b \longrightarrow \infty$$

since $I(k) \longrightarrow I$ and $I(k) - I(b) \longrightarrow 0$ by (*).

Next we have

Theorem 2. Assume that $f(x) \geq 0$ for $x \geq a$ and that it is integrable on $[a, b]$ for any $b \geq a$. Then $\int_a^\infty f(x)dx$ converges if and only if there exists M such that

$$\int_a^b f(x)dx \leq M \; forany \;\; b \geq a.$$

Proof. Since $f(x) \geq 0$ the function

$$I(b) = \int_a^b f(x)dx, \;\; a \geq b$$

is increasing. Therefore, $I(b)$ converges as $b \longrightarrow \infty$ if and only if $I(b)$ is bounded.

Example. The integral $\int_1^\infty 1x^2 + x + 1dx$ converges since

$$\int_1^b 1x^2 + x + 1dx \leq \int_1^b 1x^2 dx = 1 - 1b \leq 1 \;\; forany \;\; b \geq 1.$$

Theorem 3. (Comparison Test). Suppose that $f(x), g(x) and\, h(x)$ are integrable on $[a, b]$ for any $b \geq a$ and that $0 \leq g(x) \leq 1|f(x)| \leq h(x)$ for any $x \geq A$, where A is some number with $A \geq a$. We have the following:

a) If $\int_a^\infty h(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.

b) If $\int_a^\infty g(x)dx = \infty$ and $f(x) \geq 0$ then $\int_a^\infty f(x)dx = \infty$.

Example. The integral $\int_1^\infty \sin xxdx$ converges since for $x \geq 1$ we have $|\sin xx^2| \leq 1x^2$ and $\int_1^\infty 1x^2 dx < \infty$.

Example. The integral $\int_1^\infty \sqrt{x}2x + 1dx$ diverges since for $x \geq 1$

$$\sqrt{x}2x + 1 \geq \sqrt{x}2x + x = 131\sqrt{x}$$

and $\int_1^\infty 131\sqrt{x}$ diverges.

Proof. We will use the Cauchy criterion for $g(x)$. Let $\epsilon > 0$. Since $\int_a^\infty g(x)dx$ converge, there exists $N = N(\epsilon) \geq a$ such that

$$d \geq c \geq N \Longrightarrow \int_c^d g(x)dx < \epsilon.$$

Then for $d \geq c \geq N$ we have

$$|\int_c^d f(x)dx| \leq \int_c^d |f(x)|dx \leq \int_c^d g(x)dx < \epsilon.$$

Therefore $\int_a^\infty f(x)dx$ converges.

Example. (Gamma Function) Let $s > 0$ and

$$f(x) = x^{s-1}e^{-x}, \ \ x \in (0, \infty).$$

For $s \geq 1$ the function $f(x)$ is bounded on any interval $[0, a], \ \ a > 0$. Since

$$x^{s-1}e^{-x}e^{-12} = x^{s-1}e^{-12x} \longrightarrow 0 \ \ as \ \ x \to \infty$$

we have that there is $A > 0$ such that

$$x^{s-1}e^{-x}e^{-12x} \leq 1, \ \ for \ \ x \geq A.$$

Therefore

$$0 \leq x^{s-1}e^{-x} \leq e^{-12x}, \ \ for \ \ x \geq A.$$

By the comparison test, the integral $\int_0^\infty x^{s-1}e^{-x}dx$ converges to a finite number which we denote by $\Gamma(s)$, i.e.

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx, s \geq 1.$$

If $0 < s < 1$ then the function $f(x) = x^{s-1}e^{-x}$ is unbounded near 0. Since

$$0 \leq x^{s-1}e^{-x} \leq 1x^{1-s}, \ 0 < x \leq 1$$

and $\int_0^1 dxx^{1-s}$ converges for $0 < s < 1$ we have, by the comparison test, that $\int_0^1 x^{s-1}e^{-x}dx$ converges.

If $0 < s < 1$ and $x \geq 1$ then $0 \leq x^{s-1}e^{-x} \leq e^{-x}$ and since $\int_1^\infty e^{-x}dx$ converges, by the comparison test we conclude that $\int_1^\infty x^{s-1}e^{-x}dx$ converges. Therefore, we have that $\int_0^\infty x^{s-1}e^{-x}dx$ converges for any $s > 0$ to a finite number which is denoted by $\Gamma(s)$ and it is called the **Gamma function**, i.e.

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx, \ \ s > 0.$$

If $s > 1$, then by integrating by parts we obtain

$$\Gamma(s) = \int_0^\infty x^{s-1} d(-e^{-x}) = -x^{s-1}e^{-x}|_0^\infty + (s-1)\int_0^\infty x^{s-2}e^{-x}dx$$

Thus, if $s > n$ by integrating by parts repeatedly we obtain

$$\Gamma(s) = (s-1)(s-2)\cdots(s-n)\Gamma(s-n)$$

In particular we obtain $\Gamma(n+1) = n(n-1)\cdot 1\cdot\Gamma(1) = n!$ since $\Gamma(1) = 1$.

Example. Since $e^{-x^2} \leq e^{-x}$ for $x > 1$ the integral $\int_0^\infty e^{-x^2}dx$ converges. It can be shown that

$$\int_0^\infty e^{-x^2}dx = 12\sqrt{\pi}.$$

(See p. 371, no. 54.)

Theorem 4. (Limit Comparison Test) Suppose $f(x) \geq 0$ and $g(x) > 0$ and that they are integrable on any interval $[a, b],\ b \geq a$. If

$$\lim_{x\to\infty} f(x)g(x) = l \neq 0$$

then either both integrals

$$\int_a^\infty f(x)dx \ \ and \ \ \int_a^\infty g(x)dx$$

converge or they both diverge.

Example. The integral $\int_1^\infty 2x - 1x^5 + x^3 + xdx$ converges since

$$2x - 1x^5 + x^3 + 11x^4 = 2x^5 - x^4x^5 + x^3 + 1 \ \ @>> x \to \infty > \ \ 2 \neq 0$$

and since $\int_1^\infty 1x^4dx < \infty$.

Example. The integral

$$\int_1^\infty 1\sqrt{x^2+5x+8}dx$$

diverges since

$$1\sqrt{x^2+5x+8}1x = x\sqrt{x^2+5x+8} = 1\sqrt{1+5x+8x^2}@>> x \to \infty >1 \neq 0$$

and $\int_1^\infty 1xdx = \infty$.

<u>Proof of Theorem 4</u>. Since $\lim_{x\to\infty} f(x)g(x) = l > 0$, by the definition of the limit for $\epsilon = l2$ there exists $A = A(\epsilon)$ such that

$$x \geq A \Longrightarrow \ |f(x)g(x) - l| \ < l2.$$

The last relation is written as

$$-l2 < f(x)g(x) - l < l2, \ \ x \geq A$$

or

$$l2 < f(x)g(x) < 3l2, \ \ x \geq A$$

or

$$l2g(x) < f(x) < 3l2g(x), \ \ x \geq A$$

By Thoeorem 4 the last relation implies that

$$\int_a^\infty f(x)dx \int_a^\infty g(x)dx$$

are either both converging or both diverging.

<u>Example.</u> None of the convergence tests we discussed above can be used to show that

$$\int_1^\infty \sin xxdx$$

converges. This integral converges because of two reasons:

i) For any $b \geq 1, \ \ |\int_1^b \sin xdx| \ \ \leq 2.$

ii) The function $g(x) = 1x$ is decreasing on any interval $[1, b], \ b \geq 1.$

The convergence of the above integral follows from:

<u>Theorem 5</u> (Dirichlet's Test) If
i) f is continuous on $[a, \infty)$ and there is M such that

$$|\int_a^b f(x)dx| \leq M, \ \ foranyb \geq a$$

and ii) g is a decreasing-continuously differentiable function on $[a, \infty)$ with $g(x) \to 0 \ \ as \ \ x \to \infty$, then

$$\int_a^\infty f(x)g(x)dx$$

converges.

Remark. The condition that g is continuously differentiable can be dropped.To prove Theorem 5, we need the version of the **Second Mean Value Theorem for Integrals.** Let $f(x)$ be continuous on $[a,b]$ and $g(x)$ be decreasing and continuously differentiable on $[a,b]$. Then there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = g(a)\int_a^\xi f(x)dx + g(b)\int_\xi^b f(x)dx.$$

Proof. Let

$$F(x) = \int_a^x f(t)dt.$$

Then by the Fundamental Theorem of Calculus, we have $F'(x) = f(x)$. Integrating by parts, we obtain

$$\int_a^b f(x)g(x)dx = \int_a^b g(x)d(F(x)) = g(x)F(x)|_a^b - \int_a^b F(x)g'(x)dx.$$

Since $-g'(x) \geq 0$ on $[a,b]$ by the Mean Value Theorem for integrals we obtain that there is $\xi \in [a,b]$ such that

$$\int_a^b F(x)(-g'(x)]dx = F(\xi)\int_a^b (-g'(x))dx.$$

Therefore

$$\int_a^b f(x)g(x)dx = g(b)F(b) - g(a)F(a) - F(\xi)\int_a^b g'(x)dx$$

Proof of Theorem 5. Let $d \geq c \geq a$. By the last result there exists $\xi \in [c,d]$ such that

$$\int_c^d f(x)g(x)dx = g(c)\int_c^\xi f(x)dx + g(d)\int_\xi^d f(x)dx$$

since

$$|\int_c^\xi f(x)dx| = |\int_a^\xi f(x)dx - \int_a^c f(x)dx| \leq |\int_a^\xi f(x)dx| + |\int_a^\xi f(x)|dx \leq 2M.$$

and similarly

$$|\int_\xi^d f(x)dx| \leq 2M$$

we have that

$$|\int_c^d f(x)g(x)dx| \leq g(c)2M + g(d)2M = 4Mg(c)@>> c \to \infty > 0$$

Therefore, by the Cauchy criterion $\int_a^\infty f(x)g(x)dx$ converges.

Example. If $p > 0$ then the integral

$$\int_1^\infty \sin xx^p dx$$

converges since, if we let $f(x) = \sin x$ and $g(x) = 1x^p$, then f and g satisfy the assumptions of Theorem 5. In fact, for $p > 1$ it follows by the Comparison Test since

$$|\sin xx^p| \leq 1x^p \quad and \quad \int_1^\infty 1x^p dx < \infty.$$

Also, the integral $\int_0^\infty \sin xxdx$ converges since for $x \geq 1$ we can use the Dirichlet's test and since $\sin xx$ is continuous on $[0, 1]$. It can be shown (see p. 372, no. 55) that

$$\int_0^\infty \sin xxdx = \pi 2.$$

Remark We remark here that

$$\int_0^\infty |\sin x|xdx = \infty.$$

In fact, for any $k = 2, 3, 4, \cdots$

$$\int_0^\infty |\sin x|xdx \geq \int_\pi 2\pi|\sin x|xdx + \int_{2\pi}^{3\pi} |\sin x|xdx + \cdots + \int_{(k-1)\pi}^{k\pi} |sinx|xdx$$

Since

$$1j > \int_j^{j+1} 1xdx$$

We have

$$\int_0^\infty |\sin x|x \geq 2\pi[\int_2^3 1xdx + \int_3^4 1xdx + \cdots + \int_k^{k+1} 1xdx] \geq 2\pi \int_2^{k+1} 1xdx.$$

Thus

$$\int_0^\infty |\sin x|x \geq 2\pi \int_2^\infty 1xdx = \infty.$$

By the comparison test we have seen that if $\int_a^\infty |f(x)|dx$ converges, then so does the integral $\int f(x)dx$.

The last remark shows that the converse is not true. If $\int_a^\infty |f(x)|dx$ converges, then we say that the improper integral of f is **absolutely convergent.**

Exercises.

1. Show that $\int_0^\infty \sin(x^2)dx$ is convergent. (Hint: Let the substitution $t = x^2$.)

2. Show that the integral $\int_{-1}^\infty (1-x^2)^{-14}dx$ is convergent.

3. Compute the integral $\int_1^\infty x^3 + x^2 + 1x^6 + x^3 dx$.