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Taylor's Theorem

Theorem. If f has $n+1$ continuous derivatives on the closed interval with endpoints x_0 and x, then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + 12! f''(x_0)(x-x_0)^2 + \cdots + 1n! f^{(n)}(x_0)(x-x_0)^n + R_n(x)$$

where the remainder

$$R_n(x) = 1n! \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt, \quad (IntegralForm)$$

Also, the remainder can be written in the form

$$R_n(x) = 1(n+1)! f^{(n+1)}(\xi)(x-x_0)^{n+1}, \quad (Lagrangeremainder)$$

for some ξ between x_0 and x.

Another form of the remainder is

$$R_n(x) = 1n!(x-x_0)^{n+1} \int_0^1 f^{(n+1)}(x_0 + s(x-x_0))(1-s)^n ds.$$

Proof. By the Fundamental Theorem of Calculus we have

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt$$

Continuing till we meet $f^{(n+1)}$, we obtain

$$f(x) = f(x_0) + f'(x_0) + f'(x_0)(x-x_0) + 12! f''(x_0)(x-x_0)^2 +$$

$$\int_a^b h(t)g(t)dt = h(\xi) \int_a^b g(t)dt.$$

If $x > x_0$ then for the remainder $R_n(x)$ we apply the above result with $[a,b] = [x_0, x]$.

$$h(x) = f^{(n+1)}(t) \;\; and \;\; g(x) = (x-t)^n$$

to find $\xi \in (x_0, x)$ such that

$$R_n(x) = 1n! f^{(n+1)}(\xi) \int_{x_0}^{x} (x-t)^n dx$$

We work similarly when $x < x_0$.

Next in the integral form of the remainder we use the following change of variables:

$$s = t - x_0 x - x_0 \;\; or \;\; t = x_0 + s(x - x_0).$$

Then $dt = (x - x_0)ds$ and

$$R_n(x) = 1n! \int_0^1 f^{(n+1)}(x_0 + s(x - x_0))(x - x_0 - s(x - x_0))^n (x - x_0)ds$$

This completes the proof of the theorem.

Remark. For the Taylor's Theorem with Lagrange's remainder we can assume the weaker assumption that $f^{(n+1)}$ exists on (x_0, x) $\big(i.e., \;\; f^{(n+1)} maynotbecontinuouson(x_0, x)\big)$. The proof of this is different and can be found in the book.