MATH 166: HONORS CALCULUS II
EXAM I SOLUTIONS

Problems 1 and 2 are definitions and statements of theroems that can be found in the text.

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3. Compute $\frac{d}{dx}\int_{-\log(x)}^{\log(x)} \sin(e^t)\,dt$

Let $A(x) = \int_a^x \sin(e^t)dt$. Then $A'(x) = \sin(e^x)$. Taking the derivative of $\int_{-\log(x)}^{\log(x)} \sin(e^t)\,dt = A(\log(x)) - A(-\log(x))$ gives

$$A'(\log(x))\frac{1}{x} - A'(-\log(x))(-\frac{1}{x}) = \sin(e^{\log(x)})\frac{1}{x} + \sin(e^{-\log(x)})\frac{1}{x} = \frac{\sin(x)+\sin(1/x)}{x}$$

a) Compute $\frac{d}{dx}\cos(x)^{\sin(x)}$.

$$\begin{aligned}\frac{d}{dx}\cos(x)^{\sin(x)} &= \frac{d}{dx}\exp[\sin(x)\log(\cos(x))] = \exp[\sin(x)\log(\cos(x))][\cos(x)\log(x) - \frac{\sin^2(x)}{\cos(x)}] \\ &= \cos(x)^{\sin(x)}[\cos(x)\log(x) - \sin(x)\tan(x)]\end{aligned}$$

b) Prove $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.
$y = \arcsin(x) \Leftrightarrow \sin(y) = x$. By the Theorem on Derivatives of Inverse Functions,

$$\frac{d}{dx}\arcsin(x) = \frac{]}{\sin'(y)} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

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4. Compute the following integrals.
a) $\int \frac{x-1}{\sqrt{x-2}}\,dx$
Let $u = x-2$. Then $x-1 = u+1$ and $du = dx$, so

$$\int \frac{x-1}{\sqrt{x-2}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int u^{1/2} + u^{-1/2}\,du = \frac{2}{3}u^{3/2} + 2u^{1/2} + C = \frac{2}{3}(x-2)^{3/2} + 2\sqrt{x-2} + C$$

b) $\int x^2 e^x\,dx$
Use integration by parts twice. The first time let $u = x^2$, $dv = e^x dx$, $du = 2xdx$, $v = e^x$, so

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx$$

The second time let $u = x$, $dv = e^x dx$, $du = dx$, $v = e^x$, so

$$x^2e^x - 2\int xe^x\,dx = x^2e^x - 2[xe^x - \int e^x\,dx] = x^2e^x - 2xe^x + 2e^x + C$$

c) $\int \log x\,dx$

Ue integration by parts. Let $u = \log(x)$, $dv = dx$, $du = \frac{1}{x}\,dx$, $v = x$, so

$$\int \log x\,dx = x\log(x) - \int x\frac{1}{x}\,dx = x\log(x) - x + C$$

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d) $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

Use a trig substitution, $\tan(u) = x$, $\sec^2(u)du = dx$, $x^2+1 = \tan^2(u)+1 = \sec^2(u)$, to get

$$\begin{aligned}
\int \frac{1}{(x^2+1)^2}\,dx &= \int \frac{1}{\sec^4(u)}\sec^2(u)\,du = \int \cos^2(u)\,du \\
&= \int \frac{1}{2} + \frac{1}{2}\cos(2u)\,du = \frac{1}{2}u + \frac{1}{4}\sin(2u) + C = \frac{1}{2}u + \frac{1}{2}\sin(u)\cos(u) + C \\
&= \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{\tan(u)}{\sec^2(u)} + C = \frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)} + C
\end{aligned}$$

e) $\displaystyle\int \frac{1}{\sqrt{1+x-x^2}}\,dx$

Complete the square, $1+x-x^2 = \frac{5}{4} - (x-\frac{1}{2})^2$, and use the standard formula

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C$$

to obtain

$$\int \frac{1}{\sqrt{1+x-x^2}}\,dx = \int \frac{1}{\sqrt{\frac{5}{4}-(x-\frac{1}{2})^2}}\,dx = \arcsin\left(\frac{2}{\sqrt{5}}(x-\frac{1}{2})\right) + C = \arcsin\left(\frac{2x-1}{\sqrt{5}}\right) + C$$

f) $\displaystyle\int \frac{1}{x^2-5x+7}\,dx$

Complete the square again, $x^2-5x+7 = (x-\frac{5}{2})^2 + \frac{3}{4}$, and use the standard formula

$$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

to obtain

$$\int \frac{1}{x^2-5x+7}\,dx = \int \frac{1}{(x-\frac{5}{2})^2+\frac{3}{4}}\,dx = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}(x-\frac{5}{2})\right) + C = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x-5}{\sqrt{3}}\right) + C$$

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5. a) Give the partial fraction decomposition of $\displaystyle\frac{4x^3-x^2-4x+5}{(x-1)^2(x^2+1)}$

The decomposition should have the form

$$\frac{4x^3-x^2-4x+5}{(x-1)^2(x^2+1)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^s} + \frac{Bx+C}{x^2+1}$$

Multiplying through by the denominator on the left-hand side, we get

$$4x^3-x^2-4x+5 = A_1(x-1)(x^2+1) + A_2(x^2+1) + (Bx+C)(x^2-1)$$

Plugging in $x=1$ we see immediately that $A_2 = 2$. Equating the coefficients of x^k, $k = 3,2,1,0$, on both sides of this equation gives, respectively,

$$\begin{aligned}
4 &= A_1 + B \\
-1 &= -A_1 + A_2 - 2B + C \\
-4 &= A_1 + B - 2C \\
5 &= -A_1 + A_2 + C
\end{aligned}$$

Subtract the second equation from the fourth to get $6 = 2B$, or $B = 3$. The first equation then gives $A_1 = 4 - B = 4 - 3 = 1$, and the fourth equation gives $C = 5 + A_1 - A_2 = 5 + 1 - 2 = 4$.

b) Use a trignometric substitution to integrate $\int \frac{x}{\sqrt{x^2-6x}}\,dx$.

Complete the square, $x^2-6x=(x-3)^2-9$, and use the substitution $3\sec(u)=x-3$, $3\sec(u)\tan(u)du=dx$, $\sqrt{(x-3)^2-3^2}=3\sqrt{\sec^2(u)-1}=3\tan(u)$ to obtain

$$\begin{aligned}
\int \frac{x}{\sqrt{(x-3)^2-3^2}}\,dx &= \int \frac{3\sec(u)+3}{3\tan(u)} 3\sec(u)\tan(u)\,du \\
&= 3\int \sec^2(u)+\sec(u)\,du = 3\tan(u)+3\log|\sec(u)+\tan(u)|+C \\
&= \sqrt{x^2-6x}+3\log\left|\frac{x-3}{3}+\frac{\sqrt{x^2-6x}}{3}\right|+C
\end{aligned}$$

c) Convert to an integral involving rational functions: $\int \frac{\sin x}{2+\cos x}\,dx$.

Substitute $u=\tan(x/2)$, $dx=\frac{2}{1+u^2}\,du$, $\sin(x)=\frac{2u}{1+u^2}$ and $\cos(x)=\frac{1-u^2}{1+u^2}$ to get

$$\int \frac{\sin x}{2+\cos x}\,dx = \int \frac{\frac{2u}{1+u^2}}{\left(2+\frac{1-u^2}{1+u^2}\right)}\frac{2}{(1+u^2)}\,du = \int \frac{4u}{(1+u^2)(2(1+u^2)+1-u^2)}\,du = \int \frac{4u}{(1+u^2)(3+u^2)}\,du$$

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6. Find the Taylor polynomials.

a) $T_{4n+1}\left(\frac{x}{1+x^4}\right)$

The algebraic identity $\frac{1}{1-u}=1+u+u^2+\cdots+u^n+\frac{u^{n+1}}{1-u}$ implies

$$\begin{aligned}
\frac{x}{1+x^4} &= x\left(\frac{1}{1-(-x^4)}\right) = x\left(1+(-x^4)+(-x^4)^2+\cdots+(-x^4)^n+\frac{(-x^4)^{n+1}}{1-(-x^4)}\right) \\
&= x-x^5+x^9-\cdots+(-1)^n x^{4n+1}+x^{4n+1}\frac{(-1)^{n+1}x^{4n}}{1+x^4}
\end{aligned}$$

The last term is of the form $x^{4n+1}g(x)$ where $g(0)=0$, so our standard theorem about Taylor polynomials implies that

$$T_{4n+1}\left(\frac{x}{1+x^4}\right) = x-x^5+x^9-\cdots+(-1)^n x^{4n+1} = \sum_{k=0}^{n}(-1)^k x^{4k+1}$$

b) $T_{2n}(\cos^2(x))$

Using the identity $\cos^2(x)=\frac{1}{2}+\frac{1}{2}\cos(2x)$ and the linearity property of Taylor polynomials, we get

$$T_{2n}(\cos^2(x)) = T_{2n}\Big(\frac{1}{2}+\frac{1}{2}\cos(2x)\Big) = \frac{1}{2}+\frac{1}{2}T_{2n}(\cos(2x))$$

We know the Taylor polynomial of cosine,

$$T_{2n}(\cos(x)) = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots+\frac{(-1)^n}{(2n)!}x^{2n}$$

so by the substitution property,

$$T_{2n}(\cos(2x)) = 1-\frac{1}{2!}(2x)^2+\frac{1}{4!}(2x)^4-\cdots+\frac{(-1)^n}{(2n)!}(2x)^{2n}$$

Therefore,

$$\begin{aligned}
T_{2n}(\cos^2(x)) &= \frac{1}{2}+\frac{1}{2}(1-\frac{1}{2!}(2x)^2+\frac{1}{4!}(2x)^4-\cdots+\frac{(-1)^n}{(2n)!}(2x)^{2n}) \\
&= 1-\frac{2^1}{2!}x^2+\frac{2^3}{4!}x^4-\cdots+\frac{(-1)^n 2^{2n-1}}{(2n)!}x^{2n} \\
&= 1+\sum_{k=1}^{n}\frac{(-1)^k 2^{2k-1}}{(2k)!}x^{2k}
\end{aligned}$$