MATH 166: HONORS CALCULUS II
EXAM II SOLUTIONS

APRIL 8, 1999

Problems 1 and 2 are definitions and statements of theroems that can be found in the text.

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3. a) Integrating the formula $\frac{1}{1+x} = 1 - x + x^2 + o(x^3)$, gives $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)$ so $x^3\log(1+x) = x^4 - \frac{1}{2}x^5 + \frac{1}{3}x^6 + o(x^6)$.

b) Using the rule $\frac{1}{1-u} = 1 + u + o(u)$ with $u = x^2 - x^3 + o(x^3)$ we get

$$\begin{aligned}\frac{1+x+x^2+o(x^3)}{1-x^2+x^3+o(x^3)} &= (1+x+x^2+o(x^3))(1+x^2-x^3+o(x^3)+o(x^2-x^3+o(x^3))) \\ &= (1+x+x^2+o(x^3))(1+x^2+o(x^2)) = 1+x+2x^2+o(x^2)\end{aligned}$$

c) The Lagrange form of the remainder is $E_nf(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)x^{n+1}$ for some c in $[-1,1]$ (between 0 and x). Since $f(x) = \sin(x)$, the derivatives of $f(x)$ are either $\pm\sin(x)$ or $\pm\cos(x)$, so $|f^{(n+1)}(x)| \le 1$. Thus, $|E_nf(x)| \le \frac{1}{(n+1)!}|x|^{n+1} \le \frac{1}{(n+1)!}$ since, by assumption, $|x| < 1$. Therefore $|E_n| < 10^{-3}$ if $(n+1)! > 1000$ which occurs for $n \ge 6$ ($6! = 720$ and $7! = 5040$).

d) By a familiar algebraic identity, the partial sums are $s_n = \sum_{k=0}^{n} r^k = \frac{1}{1-r} - \frac{r^{n+1}}{1-r}$. Since $|r|^{n+1} \to 0$ as $n \to \infty$ when $|r| < 1$, the limit of the partial sums is $\lim_{n\to\infty} s_n = \frac{1}{1-r}$.

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4. a) $\lim_{x\to0}(1+h(x))^{1/h(x)} = \exp[\lim_{x\to0}\log(1+h(x))/h(x)] = \exp[\lim_{x\to0}\frac{h'(x)/(1+h(x))}{h'(x)}] = \exp[\lim_{x\to0}\frac{1}{1+h(x)}] = e$. We could also solve this problem assuming only that $h(x)$ is continuous by substituting $u = h(x)$ and $u \to 0$ for $x \to 0$: $\exp[\lim_{x\to0}\log(1+h(x))/h(x)] = \exp[\lim_{u\to0}\log(1+u)/u] = \exp[\lim_{u\to0}\frac{1}{1+u}] = e$ (use L'Hôpital's rule, taking derivatives with respect to the variable u!).

b) Use L'Hopital's rule twice: $\lim_{x\to0}\frac{e^{x^2}-\cos(x)}{x^2} = \lim_{x\to0}\frac{2xe^{x^2}+\sin(x)}{2x} = \lim_{x\to0}\frac{(2+4x^2)e^{x^2}+\cos(x)}{2} = \frac{3}{2}$. We could also use o-notation: $\lim_{x\to0}\frac{e^{x^2}-\cos(x)}{x^2} = \lim_{x\to0}\frac{(1+x^2+o(x^3))-(1-\frac{1}{2}x^2+o(x^3)}{x^2} = \lim_{x\to0}\frac{3}{2}+o(x) = \frac{3}{2}$.

c) $\lim_{x\to\infty}\sqrt{2x+x^2}-\sqrt{x+x^2} = \lim_{x\to\infty}\frac{(2x+x^2)-(x+x^2)}{\sqrt{2x+x^2}+\sqrt{x+x^2}} = \lim_{x\to\infty}\frac{1}{\sqrt{\frac{2}{x}+1}+\sqrt{\frac{1}{x}+1}} = \frac{1}{2}$. We could also substitute $x = \frac{1}{y}$: $\lim_{x\to\infty}\sqrt{2x+x^2}-\sqrt{x+x^2} = \lim_{y\to0^+}\frac{\sqrt{2y+1}-\sqrt{y+1}}{y} = \lim_{y\to0^+}\frac{(2y+1)-(y+1)}{y(\sqrt{2y+1}+\sqrt{y+1})} = \lim_{y\to0^+}\frac{1}{\sqrt{2y+1}+\sqrt{y+1}} = \frac{1}{2}$.

d) $\lim_{n\to\infty}\frac{(n+(-1)^n)(n+1)}{3n^2} = \lim_{n\to\infty}\frac{n^2+(1+(-1)^n)n+1}{3n^2} = \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1+(-1)^n}{3n}+\frac{1}{3n^2}\right) = \frac{1}{3}$

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5. a) $\displaystyle\sum_{n=0}^{\infty}\frac{a^n+(-1)^n}{b^{2n}}=\sum_{n=0}^{\infty}\left(\frac{a}{b^2}\right)^n+\left(\frac{-1}{b^2}\right)^n=\frac{1}{1-a/b^2}+\frac{1}{1+1/b^2}$ (linearity and geometric series).

b) $\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{(n+1)^2}=\frac{1}{1^2}-\lim_{n\to\infty}\frac{1}{(n+1)^2}=1$ (telescoping sum).

c) $\displaystyle\sum_{n=0}^{\infty}\frac{n}{n+1}x^n=\sum_{n=0}^{\infty}\left(1-\frac{1}{n+1}\right)x^n=\sum_{n=0}^{\infty}x^n-\sum_{n=0}^{\infty}\frac{1}{n+1}x^n=\frac{1}{1-x}-\frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}$
$\displaystyle=\frac{1}{1-x}-\frac{1}{x}\int_0^x\sum_{n=0}^{\infty}t^n\,dt=\frac{1}{1-x}-\frac{1}{x}\int_0^x\frac{1}{1-t}\,dt=\frac{1}{1-x}+\frac{1}{x}\log(1-x)$. We could also do the following:
$\displaystyle\sum_{n=0}^{\infty}\frac{n}{n+1}x^n=x\frac{d}{dx}\sum_{n=0}^{\infty}\frac{1}{n+1}x^n=x\frac{d}{dx}\left[\frac{1}{x}\int_0^x\sum_{n=0}^{\infty}t^n dt\right]=x\frac{d}{dx}\left[\frac{1}{x}\int_0^x\frac{1}{1-t}dt\right]=x\frac{d}{dx}\left[-\frac{1}{x}\log(1-x)\right]=$
$\displaystyle x[\frac{1}{x^2}\log(1-x)+\frac{1}{x(1-x)}]=\frac{1}{x}\log(1-x)+\frac{1}{1-x}$

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6. a) Since $\displaystyle\frac{1}{n!}<\frac{1}{2^{n-1}}$ for integers $n\geq 1$, and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ converges (geometric series), the Comparison Test implies that $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}$ converges. We could also compare $\displaystyle\frac{1}{n!}\leq\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$ and $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ converges as a telescoping sum.

b) First we note that $\displaystyle\frac{\sqrt{n+1}}{n^2}$ is asymptotically equal to $\displaystyle\frac{1}{n^{3/2}}$: $\displaystyle\lim_{n\to\infty}\frac{\sqrt{n+1}/n^2}{1/n^{3/2}}=\lim_{n\to\infty}\sqrt{1+\frac{1}{n}}=1$. We could also compare $\displaystyle\frac{\sqrt{n+1}}{n^2}\leq\frac{\sqrt{2n}}{n^2}=\frac{\sqrt{2}}{n^{3/2}}$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges (by the Integral Test, for example), the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n+1}}{n^2}$ converges.

c) Using the substitution $u=\log(x)$, $du=\frac{1}{x}\,dx$, we find

$$\int_2^n\frac{1}{x\log(x)}\,dx=\int_{\log(2)}^{\log(n)}\frac{1}{u}\,du=\log|\log(n)|-\log|\log(2)|\to\infty\text{ as }n\to\infty$$

Therefore the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\log(n)}$ diverges by the Integral Test.
