MATH 166: HONORS CALCULUS II
FINAL EXAM SOLUTIONS

Problems 1 and 2 are definitions and theroems that can be found in the text.

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3. a) $y = \arctan(x) \Leftrightarrow \tan(y) = x$. $\dfrac{dy}{dx} = \dfrac{1}{dx/dy} = \dfrac{1}{\sec^2(y)} = \dfrac{1}{1+\tan^2(y)} = \dfrac{1}{1+x^2}$.

b) If $f(x) = \dfrac{x+1}{x-1}$ then $f'(x) = -2/(x-1)^2$. Thus, $f(x)$ is strictly decreasing on the intervals $x < 1$ and $x > 1$ and therefore has an inverse on each of those intervals. To find $f^{-1}(x)$, solve $y = \dfrac{x+1}{x-1}$ for x: $yx - y = x + 1$, $x(y-1) = y+1$ and $x = \dfrac{y+1}{y-1}$. Therefore, $f^{-1}(x) = \dfrac{x+1}{x-1} = f(x)$ (!).

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4. a) Let $A(x) = \displaystyle\int_0^x \exp(1-t^2)\,dt$, so $A'(x) = \exp(1-x^2)$. Then $\displaystyle\int_{\sin(x)}^{\cos(x)} \exp(1-t^2)\,dt = A(\cos(x)) - A(\sin(x))$ and by the chain rule $\dfrac{d}{dx}\displaystyle\int_{\sin(x)}^{\cos(x)} \exp(1-t^2)\,dt = A'(\cos(x))(-\sin(x)) - A'(\sin(x))\cos(x) = -\sin(x)\exp(1-\cos^2(x)) - \cos(x)\exp(1-\sin^2(x)) = -\sin(x)\exp(\sin^2(x)) - \cos(x)\exp(\cos^2(x))$.

b) Let $f(x) = x^{-x}$, $x > 0$. Since $f'(x) = \exp(-x\log(x))[-\log(x) - 1]$, we see that $f(x)$ is increasing ($f'(x) > 0$) if $\log(x) + 1 < 0$ and decreasing ($f'(x) < 0$) if $\log(x) + 1 > 0$. This implies that $f(x)$ has a maximum when $\log(x) + 1 = 0$, i.e., at $x = e^{-1} = 1/e$ with maximum value $f(1/e) = e^{1/e}$.

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5. a) Substituting $u = x+4$, $du = dx$ gives $\displaystyle\int \frac{x}{\sqrt{x+4}}\,dx = \int u^{1/2} - 4u^{-1/2} du = \frac{2}{3}(x+4)^{3/2} - 8(x+4)^{1/2} + C = \frac{2}{3}(x-16)\sqrt{x+4} + C$.

b) $\displaystyle\int \frac{1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{1}{x} - \frac{1}{x+2}\,dx = \frac{1}{2}(\log|x| - \log|x+2|) + C$.

c) $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x+1)^2}}\,dx = \arcsin((x+1)/2) + C$.

d) Substituting $u = \tan(\theta/2)$, $d\theta = \dfrac{2du}{1+u^2}$, $\cos(\theta) = \dfrac{1-u^2}{1+u^2}$ gives $\displaystyle\int \frac{1}{2+\cos(\theta)}\,d\theta = \int \frac{1}{2+\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}\,du = \int \frac{2}{3+u^2}\,du = \frac{2}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(\theta/2)}{\sqrt{3}}\right) + C$

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e) $\int_0^\infty e^{-x}\sin(x)\,dx = \lim_{a\to\infty}\int_0^a e^{-x}\sin(x)\,dx$. Integrating by parts with $u = \sin(x)$, $dv = e^{-x}dx$ gives $\int_0^a e^{-x}\sin(x)\,dx = -\sin(a)e^{-a} + \int_0^a \cos(x)e^{-x}\,dx$. Repeating this with $u = \cos(x)$ gives $\int_0^a e^{-x}\sin(x)\,dx = -\sin(a)e^{-a} - \cos(a)e^{-a} + 1 - \int_0^a \sin(x)e^{-x}\,dx$, so $\int_0^a e^{-x}\sin(x)\,dx = \frac{1}{2} - \frac{1}{2}e^{-a}(\sin(a)+\cos(a))$. Therefore, in the limit as $a \to \infty$ we get $\int_0^\infty e^{-x}\sin(x)\,dx = 1/2$.

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6. a) $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{n!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

b) $\frac{x}{x^2+3} = \frac{x}{3}\Big(\frac{1}{1-(-x^2/3)}\Big) = \frac{x}{3}\Big(1 + \Big(\frac{-x^2}{3}\Big) + \Big(\frac{-x^2}{3}\Big)^2 + \cdots + \Big(\frac{-x^2}{3}\Big)^n + \frac{(-x^2/3)^{n+1}}{1+(x^2/3)}\Big)$, so
$T_{2n+1}\Big(\frac{x}{x^2+3}\Big) = \frac{x}{3} - \frac{x^3}{3^2} + \frac{x^5}{3^3} - \cdots + \frac{(-1)^n x^{2n+1}}{3^{n+1}}$

c) Since $\log(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 + o(u^3)$, $\log(1+x^2) - \log(1+x^3) = [x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 + o(x^7)] - [x^3 - \frac{1}{2}x^6 + o(x^8)] = x^2 - x^3 - \frac{1}{2}x^4 + \frac{5}{6}x^6 + o(x^7)$. Hence $T_7[\log(1+x^2) - \log(1+x^3)] = x^2 - x^3 - \frac{1}{2}x^4 + \frac{5}{6}x^6$.

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7. a) $\frac{\cos(x^2)-1}{x^2\sin^2(x)} = \frac{(1 - x^4/2 + o(x^7)) - 1}{x^2(x+o(x^2))^2} = \frac{-x^4/2 + o(x^7)}{x^2(x^2+o(x^3))} = \frac{-1/2 + o(x^3)}{1+o(x)} \to -\frac{1}{2}$ as $x \to 0$.

b) $\lim_{x\to 0^+} \frac{1}{x}\Big(\frac{1}{\sqrt{1-x}} - \frac{1}{\sqrt{1+x}}\Big) = \lim_{x\to 0^+} \frac{1}{x}\Big(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1-x}\sqrt{1+x}}\Big)$
$= \lim_{x\to 0^+} \frac{(1+x)-(1-x)}{x\sqrt{1-x^2}(\sqrt{1-x}+\sqrt{1+x})} = \lim_{x\to 0^+} \frac{2}{\sqrt{1-x^2}(\sqrt{1-x}+\sqrt{1+x})} = \frac{2}{1(1+1)} = 1$

c) $\lim_{x\to\infty} \frac{\sinh(1+x)}{\cosh(x)} = \lim_{x\to\infty} \frac{\frac{1}{2}(e^{1+x} - e^{-1-x})}{\frac{1}{2}(e^x + e^{-x})} = \lim_{x\to\infty} \frac{e^{1+x}(1 - e^{-2-2x})}{e^x(1+e^{-2x})} = \lim_{x\to\infty} \frac{e(1-e^{-2-2x})}{1+e^{-2x}} = e.$

d) $\lim_{n\to\infty} \Big(n^{100} + (-1)^n\Big)^{1/n} = \lim_{n\to\infty} n^{100/n}\Big(1 + \frac{(-1)^n}{n^{100}}\Big)^{1/n} = \lim_{n\to\infty} \exp[100\log(n)/n] \lim_{n\to\infty}\Big(1 + \frac{(-1)^n}{n^{100}}\Big)^{1/n} = \exp[0](1+0)^0 = 1$

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8. a) $\sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+2}\Big) = \sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2}\Big) = \sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+1}\Big) + \sum_{n=1}^{\infty}\Big(\frac{1}{n+1} - \frac{1}{n+2}\Big) = \Big(\frac{1}{1} - \lim_{n\to\infty}\frac{1}{n+1}\Big) + \Big(\frac{1}{1+1} - \lim_{n\to\infty}\frac{1}{n+2}\Big) = 1 + \frac{1}{2} = \frac{3}{2}$ [telescoping sums].

b) $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{3^n} = 2\sum_{n=1}^{\infty}\Big(\frac{1}{3}\Big)^n + \sum_{n=1}^{\infty}\Big(\frac{-1}{3}\Big)^n = 2\frac{1/3}{1-(1/3)} + \frac{-1/3}{1-(-1/3)} = 1 - \frac{1}{4} = \frac{3}{4}.$

c) $2x^3 + 4x^5 + 6x^7 + 8x^9 + \cdots = x^2(2x + 4x^3 + 6x^5 + 8x^7 + \cdots) = x^2 \dfrac{d}{dx}(x^2 + x^4 + x^6 + x^8 + \cdots) = x^2 \dfrac{d}{dx}\Big[\dfrac{x^2}{1-x^2}\Big] = x^2 \dfrac{2x(1-x^2) - (-2x)x^2}{(1-x^2)^2} = \dfrac{2x^3}{(1-x^2)^2}$.

d) The series $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$ converges by Leibniz' Rule. Since $\frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{6}x^6 - \frac{1}{8}x^8 + \cdots = \int_0^x (t - t^3 + t^5 - t^7 + \cdots)\,dt = \int_0^x t(1 - t^2 + t^4 - t^6 + \cdots)\,dt = \int_0^x \frac{t}{1+t^2}\,dt = \frac{1}{2}\log(1+x^2)$, Abel's Theorem implies that $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \lim_{x\to 1^-} \frac{1}{2}\log(1+x^2) = \frac{1}{2}\log(2)$

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9. a) Since $s > 1$, $\log(a)^{1-s} \to 0$ as $a \to \infty$, so $\int_2^\infty \frac{1}{x\log(x)^s}\,dx = \frac{1}{1-s}\log(x)^{1-s}\Big|_2^\infty < \infty$. By the Integral Test, the absolute value series $\sum_{n=1}^\infty \frac{1}{n\log(n)^s}$ converges. Therefore the series $\sum_{n=1}^\infty \frac{(-1)^n}{n\log(n)^s}$ is absolutely convergent.

b) Since $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$, the terms of the series $\sum_{n=1}^\infty (-1)^n\left(1 + \frac{1}{n}\right)^n$ do not approach 0, so the series diverges by the Simple Divergence Test.

c) First observe that $\frac{n}{n^2+1} \approx \frac{1}{n}$ since $\lim_{n\to\infty} \frac{n}{n^2+1}\cdot\frac{n}{1} = 1$. The harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ diverges (e.g., by the Integral Test), so the absolute value series $\sum_{n=1}^\infty \frac{n}{n^2+1}$ diverges. However, the terms $\frac{n}{n^2+1}$ strictly decrease to zero, so the series $\sum_{n=1}^\infty \frac{(-1)^n n}{n^2+1}$ converges by Leibniz' Rule and is therefore conditionally convergent.

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10. a) Applying the Ratio Test to the absolute value of the terms of the series $\sum_{n=0}^\infty (1 + (-3)^n)x^n$ yields $\lim_{n\to\infty} \frac{|1+(-3)^{n+1}||x|^{n+1}}{|1+(-3)^n||x|^n} = \lim_{n\to\infty} \frac{|1/(-3)^n + (-3)|}{|1/(-3)^n + 1|}|x| = \frac{|0+(-3)|}{|0+1|}|x| = 3|x|$. The series thus converges for $|x| < 1/3$ and diverges for $|x| > 1/3$. At the endpoints $x = \pm 1/3$, the series $\sum_{n=0}^\infty \frac{1+(-3)^n}{(\pm 3)^n}$ is divergent since its terms do not approach 0. Therefore, the interval of convergence is $(-1/3, 1/3)$.

b) Applying the Ratio Test to the absolute value of the terms of the series $\sum_{n=1}^\infty \frac{(x+1)^{2n}}{n3^n}$ yields $\lim_{n\to\infty} \frac{|x+1|^{2n+2}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{|x+1|^{2n}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\frac{1}{3}|x+1|^2 = \frac{1}{3}|x+1|^2$. The series thus converges if $\frac{1}{3}|x+1|^2 < 1$, i.e., $|x+1| < \sqrt{3}$, and diverges if $\frac{1}{3}|x+1|^2 > 1$, i.e., $|x+1| > \sqrt{3}$. At the endpoints $x + 1 = \pm\sqrt{3}$, the series is $\sum_{n=1}^\infty \frac{(\pm\sqrt{3})^{2n}}{n3^n} = \sum_{n=1}^\infty \frac{1}{n}$ which diverges (the harmonic series). Therefore the interval of convergence is $(-1-\sqrt{3}, -1+\sqrt{3})$.