## INFINITE EXPONENTIATION

MATH 166: HONORS CALCULUS II

We can make sense of an infinite exponential, $x^{x^{x^{\ldots}}}$, using a recursive definition:

$$
f_{1}(x)=x, \quad f_{n}(x)=x^{f_{n-1}(x)}
$$

The infinite exponential function $f(x)$ can then be defined to be the limit of $f_{n}(x)$ as $n$ goes to infinity. This limit will not exist for every $x$. For example, $f_{n}(1)=1$ for all $n$ so $f(1)=1$, but $f_{n}(2)$ would quickly go to infinity as $n$ increases. To see which values might work, we'll first plot and compare a few of the functions $f_{n}(x)$.

There is some unusual behavior near 0 and the graphs split apart around $x=1.4$, but the different functions $f_{n}(x)$ agree closely between these numbers. The key to understanding $f(x)$ is to realize that if this limit exists then the recursive definition $f_{n}(x)=x^{f_{n-1}(x)}$ implies $f(x)=x^{f(x)}$. Let $y=f(x)$. Then $y=x^{y}$ and we can solve: $x=y^{1 / y}$. The function $y^{1 / y}$ is the inverse of $f(x)$, and vice-versa, at least on intervals containing 1 where these functions are $1-1$. We can easily check for such intervals for the function $y^{1 / y}$.

Since $y^{1 / y}=e^{\log (y) / y}$, we compute the derivative to be

$$
\frac{d}{d y}\left[y^{1 / y}\right]=y^{1 / y}\left[-\frac{\log (y)}{y^{2}}+\frac{1}{y^{2}}\right]=y^{1 / y-2}(1-\log (y))
$$

which confirms that $y^{1 / y}$ is strictly increasing for $1-\log (y) \geq 0$, i.e., $0<y \leq e$ (this interval contains 1) and strictly decreasing for $y>e$. Therefore the desired inverse $f(x)$ exists for $0<x<e^{1 / e}=1.44467$.

The plots confirm that the graphs of $f_{n}(x)$ split apart near $x=1.4$ :

The graphs of $f_{n}(x)$ seem to split apart near 0 . When $n$ is odd, $f_{n}(x)$ goes to 0 and when $n$ is even $f_{n}(x)$ goes to 1 . However, as $n$ increases, these functions do approach a common limit: the "even" functions shrink down and the "odd" functions creep up.

We can obtain the graph of the inverse of $y^{1 / y}$ by plotting it parametrically (in Mathematica, ParametricPlot $[\{\mathrm{y}(1 / \mathrm{y}), \mathrm{y}\},\{\mathrm{y}, 0, \mathrm{E}\}])$. This plot agrees closely to the plots of $f_{n}(x)$ above.

This analysis proves that $f(x)$ exists for $0<x<e^{1 / e}$. More work would have to be done to prove that the limit of $f_{n}(x)$ as $n$ goes to infinity does not exist when $x>e^{1 / e}$. Just because an inverse to $y^{1 / y}$ cannot be defined for $y>e$ does not mean that the limit does not exist. The graphs do indicate, however, that $f_{n}(x)$ goes to infinity as $n$ goes to infinity if $x>e^{1 / e}$.

