INTEGRABILITY ON SUBINTERVALS

MATH 166: CALCULUS II

The purpose of this note is to fill in a gap in Apostol's Calculus text. The following theorem is very useful in practice and simplifies the statements of several theorems in the text.

Theorem. *Let f be a function defined on an interval $[a, b]$. If f is integrable on $[a, b]$, then f is integrable on any subinterval $[c, d] \subset [a, b]$.*

Proof. We shall prove the logically equivalent statement: If there exists a subinterval $[c, d] \subset [a, b]$ on which f is not integrable, then f is not integrable on the larger interval $[a, b]$.

By the definition of integrability, to say that f is not integrable on $[c, d]$ means that there is a constant $M > 0$ such that for any step functions s and t on $[c, d]$ satisfying $s(x) \leq f(x) \leq t(x)$ for $c \leq x \leq d$, the difference of the corresponding integrals is at least M:

$$\int_c^d s(x) - \int_c^d t(x) \geq M$$

Now let $\hat{t}$ and $\hat{s}$ be any step functions on $[a, b]$ satisfying $\hat{s}(x) \leq f(x) \leq \hat{t}(x)$ for $a \leq x \leq b$. Note that the restrictions of $\hat{s}$ and $\hat{t}$ to any subinterval of $[a, b]$ and will still be step functions on that subinterval. Moreover, these step functions are integrable on any subinterval (as are all step functions), and by additivity of intervals for step functions:

$$\int_a^b \hat{s} - \int_a^b \hat{t} = \int_a^c (\hat{s} - \hat{t}) + \int_c^d (\hat{s} - \hat{t}) + \int_d^b (\hat{s} - \hat{t})$$

All the integrals on the right hand side are non-negative since $\hat{s} \geq \hat{t}$. However, by our assumption above, the middle integral on the right hand side is at least M. Therefore,

$$\int_a^b \hat{s} - \int_a^b \hat{t} \geq M$$

Since this holds for any step functions $\hat{s} \leq f \leq \hat{t}$ on $[a, b]$, we conclude that f is not integrable on $[a.b]$.

□

Date: January 13, 1999.