MATH 166: HONORS CALCULUS II
EXAM I SOLUTIONS

1. a) $\log(x) = \int_1^x \frac{1}{t}\,dt$; domain $x > 0$, range all $y \in \mathbf{R}$.
b) $\exp(x)$ is the inverse of $\log(x)$: $y = \exp(x)$ if and only if $\log(y) = x$; domain all $x \in \mathbf{R}$, range $y > 0$.
c) $a^x = \exp(x\log(a))$; domain all $x \in \mathbf{R}$, range $y > 0$.
d) $\sinh(x) = (e^x - e^{-x})/2$; domain all $x \in \mathbf{R}$, range all $y \in \mathbf{R}$.
$\cosh(x) = (e^x + e^{-x})/2$; domain all $x \in \mathbf{R}$, range all $y \geq 1$.
e) $T_n f(x;a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

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2. a) Let f be integrable on $[a,b]$ and let $A(x) = \int_1^x f(t)\,dt$ for $x \in [a,b]$. If $f(x)$ is continuous at x, then $A(x)$ is differentiable at x and $A'(x) = f(x)$.
b) $A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt.$
Since f is continuous on the interval from x to $x+h$, the Mean Value Theorem implies there exists a c_h between x and $x+h$ such that $f(c_h) = \frac{1}{h}\int_x^{x+h} f(t)\,dt$. Therefore, using the fact that f is continuous at x and $c_h \to x$ as $h \to 0$, we get $A'(x) = \lim_{h\to 0} f(c_h) = f(\lim_{h\to 0} c_h) = f(x)$.

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3. a) Assume that f has continuous derivatives to order $n+1$ on an interval conatining a. Then,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt$$

b) If $F(t) = T_n f(x;t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$, then $F(x) = f(x)$ and $F(a) = T_n f(x;a)$, so $E_n f(x) = f(x) - T_n(x;a) = F(x) - F(a) = \int_a^x F'(t)\,dt$. Now, by the Product Rule, $F'(t) = f'(t) + \sum_{k=1}^{n}(a_{k+1} - a_k)$ where $a_k = \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}$. Since this is a telescoping sum, we get $F'(t) = f'(t) + a_{n+1} - a_1 = \frac{f^{(n+1)}(t)}{n!}(x-t)^n$, so that $E_n f(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt$.

4. a) Let $A(x) = \int_a^x \log(\cos(\sqrt{t}))\,dt$. Then by the Fundamental Theorem of Calculus, $A'(x) = \log(\cos(\sqrt{x}))$. Since $\int_{1/x^2}^{x^2} \log(\cos(\sqrt{t}))\,dt = A(x^2) - A(1/x^2)$, the Chain Rule implies

$$\begin{aligned} \frac{d}{dx}\int_{1/x^2}^{x^2} \log(\cos(\sqrt{t}))\,dt &= A'(x^2)2x - A'(1/x^2)(-2/x^3) \\ &= 2x\log(\cos(|x|)) + \frac{2}{x^3}\log(\cos(1/|x|)) = \log\left(\cos(|x|)^{2x}\cos(1/|x|)^{2/x^3}\right) \end{aligned}$$

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b)

$$\begin{aligned}\frac{d}{dx}\left(1+x^3\right)^{(1+x^5)} &= \frac{d}{dx}\exp[(1+x^5)\log(1+x^3)]\\ &= \exp[(1+x^5)\log(1+x^3)]\Big(5x^4\log(1+x^3)+\frac{1+x^5}{1+x^3}3x^2\Big)\\ &= \left(1+x^3\right)^{x^5}\Big(5x^4(1+x^3)\log(1+x^3)+3x^2(1+x^5)\Big)\end{aligned}$$

c) $f^{-1}(1)=x$ if and only if $1=f(x)=x+e^x$, and this equations holds only for $x=0$. By the Theorem on the Derivatives of Inverses,

$$(f^{-1})'(1)=\frac{1}{f'(0)}=\frac{1}{1+e^0}=\frac{1}{2}$$

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5 a) Substitute $u=\sqrt{x-1}$, $u^2+1=x$, $2u\,du=dx$ to get

$$\begin{aligned}\int\frac{\sqrt{x-1}}{x}\,dx &= \int\frac{2u^2}{u^2+1}\,du\\ &= \int 2-\frac{2}{u^2+1}\,du=2u-2\arctan(u)+C\\ &= 2\sqrt{x-1}-2\arctan(\sqrt{x-1})+C\end{aligned}$$

b) $\displaystyle\int(x\cos(x))^2\,dx = \int x^2(\frac{1}{2}+\frac{1}{2}\cos(2x))\,dx = \frac{1}{6}x^3+\frac{1}{2}\int x^2\cos(2x)\,dx$. Use integration by parts with $u=x^2$, $dv=\cos(2x)\,dx$, $du=2x\,dx$, $v=\dfrac{1}{2}\sin(2x)\,dx$ to get

$$\int x^2\cos(2x))\,dx=\frac{1}{2}x^2\sin(2x)-\int x\sin(2x)\,dx$$

Integration by parts again gives $\displaystyle\int x\sin(2x)\,dx=-\frac{1}{2}x\cos(2x)+\frac{1}{4}\sin(2x)+C$, so

$$\int(x\cos(x))^2\,dx=\frac{1}{6}x^3+\frac{1}{4}x^2\sin(2x)+\frac{1}{4}x\cos(2x)-\frac{1}{8}\sin(2x)+C$$

c) Complete the square to get $\sqrt{x^2-x}=\sqrt{(x-1/2)^2-(1/2)^2}$. Then make a trignonometric subsitution, $(1/2)\tan(u)=\sqrt{(x-1/2)^2-(1/2)^2}$, $(1/2)\sec(u)=x-1/2$, $(1/2)\sec(u)\tan(u)\,du=dx$, $x=(1/2)(\sec(u)+1)$, to get

$$\begin{aligned}\int\frac{x}{\sqrt{x^2-x}}\,dx &= \frac{1}{2}\int\frac{\sec(u)+1}{\tan(u)}\sec(u)\tan(u)\,du\\ &= \frac{1}{2}\int\sec^2(u)+\sec(u)\,du\\ &= \frac{1}{2}\tan(u)+\frac{1}{2}\log|\sec(u)+\tan(u)|+C\\ &= \sqrt{x^2-x}+\frac{1}{2}\log|2x-1+2\sqrt{x^2-x}|+C\end{aligned}$$

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6. a) $\dfrac{1}{x^3-1}=\dfrac{1}{(x-1)(x^2+x+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}$. Clear the denominators to get

$$1=A(x^2+x+1)+(Bx+C)(x-1)=(A+B)x^2+(A-B+C)x+(A-C)$$

This equation must hold for all x, so

$$\begin{array}{ccccccc} A & + & B & & & = & 0\\ A & - & B & + & C & = & 0\\ A & & & - & C & = & 1\end{array}$$

Solving these equations yields $A=1/3$, $B=-1/3$ and $C=-2/3$.

b) Use the substitution $u = \tan(x/2)$, $dx = 2du/(1+u^2)$, $\sin(x) = 2u/(1+u^2)$ and $\cos(x) = (1-u^2)/(1+u^2)$ to get

$$\int \frac{(\sin(x)+1)^2}{\cos(x)+3}\,dx = \int \frac{(2u/(1+u^2)+1)^2}{((1-u^2)/(1+u^2)+3)}\frac{2}{(1+u^2)}\,du = \int \frac{(u+1)^4}{(u^2+2)(u^2+1)^2}\,du$$

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7. a) $\tan(0) = 0$, $\tan'(0) = \sec^2(0) = 1$, $\tan''(0) = 2\sec^2(0)\tan(0) = 0$, $\tan'''(0) = 4\sec^2(0)\tan(0) + 2\sec^4(0) = 2$, so $T_3\tan(x) = x + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$.

b) First recall that $T_n\log(1-x) = T_n\Big(-\int_0^x \frac{1}{1-t}\,dt\Big) = -\int_0^x T_{n-1}\Big(\frac{1}{1-t}\Big)\,dt = -\int_0^x 1+t+\cdots+t^{n-1}\,dt = -\big(x+\frac{1}{2}x^2+\cdots+\frac{1}{n}x^n\big)$. Then by the Substitution Property, $T_n\log(1+x) = x - \frac{1}{2}x^2 + \cdots + (-1)^{n+1}\frac{1}{n}x^n$. Therfore, by the Linearity Property,

$$\begin{aligned}
T_{2n}\log(1-x^2) &= T_{2n}(\log(1-x)+\log(1+x)) = T_{2n}\log(1-x) + T_{2n}\log(1+x)\\
&= -\big(x+\frac{1}{2}x^2+\cdots+\frac{1}{2n}x^{2n}\big) + x - \frac{1}{2}x^2 + \cdots + (-1)^{2n+1}\frac{1}{2n}x^{2n}\\
&= -x^2 - \frac{1}{2}x^4 - \cdots - \frac{1}{n}x^{2n}
\end{aligned}$$

c)

$$\begin{aligned}
\frac{1+x^n}{1+x^{2n}} &= \frac{1}{1+x^n} + x^n\frac{1}{1+x^{2n}}\\
&= 1 - x^{2n} + \frac{x^{4n}}{1+x^{2n}} + x^n\Big(1 - x^{2n} + \frac{x^{4n}}{1+x^{2n}}\Big)\\
&= 1 + x^n - x^{2n} - x^{3n} + x^{3n}\frac{x^n+x^{2n}}{1+x^{2n}}
\end{aligned}$$

By the Uniqueness Theorem for Taylor Polynomials, $T_{3n}\Big(\frac{1+x^n}{1+x^{2n}}\Big) = 1 + x^n - x^{2n} - x^{3n}$.

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8. By Taylor's Formula, $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{e^{c_x}}{4!}x^4$ for some c_x between 0 and x. Then

$$\begin{aligned}
\int_{0.5}^{1.0}\frac{e^x}{x}\,dx &= \int_{0.5}^{1.0}\frac{1}{x} + 1 + \frac{1}{2}x + \frac{1}{3!}x^2 + \frac{e^{c_x}}{4!}x^3\,dx\\
&= \log(x) + x + \frac{1}{4}x^2 + \frac{1}{18}x^3\Big|_{0.5}^{1.0} + \int_{0.5}^{1.0}\frac{e^{c_x}}{4!}x^3\,dx\\
&= \log(2) + \frac{1}{2} + \frac{3}{16} + \frac{7}{144} + E\\
&\cong 1.429258 + E
\end{aligned}$$

Here E is the error term and can be estimated using the inequality $e^{c_x} < e^{1.0} = e$:

$$E = \int_{0.5}^{1.0}\frac{e^{c_x}}{4!}x^3\,dx < \int_{0.5}^{1.0}\frac{e}{4!}x^3\,dx = \frac{e}{4\cdot 4!}x^4\Big|_{0.5}^{1.0} = \frac{5e}{512} \cong 0.026546$$

Since E is positive, the value of the original integral lies between 1.429258 and $1.429258 + 0.026546 = 1.455804$. Another way to organize this information is to take the approximation to be the midpoint, 1.443531, and say that the error is $\pm 0.026545/2 = \pm 0.013273$. (The actual value of the integral is approximately 1.440898.)