Math 166: Honors Calculus II

Final Exam Solutions

Problems 1 and 2 are definitions and theroems that can be found in the text or class notes.

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3. a) $\dfrac{d}{dx}\displaystyle\int_{\log(x)}^{\tan(x)} \exp(t\sqrt{1+t^2})\,dt = \exp(\tan(x)\sqrt{1+\tan^2(x)})\sec^2(x) - \exp(\log(x)\sqrt{1+\log^2(x)})\frac{1}{x}$

b) $f(x) = x^{1/x} = e^{\log(x)/x}$ is increasing when $f'(x) = e^{\log(x)/x}\Big(\dfrac{1}{x^2} - \dfrac{\log(x)}{x^2}\Big) > 0$, i.e., when $1 - \log(x) > 0$ or $x < e$, and decreasing ($f'(x) < 0$) when $1 - \log(x) < 0$ or $x > e$. Therefore, $f(e) = e^{1/e}$ is the maximum of $f(x)$.

c) $y = \arcsin(x) \Leftrightarrow \sin(y) = x$. The formula for derivatives of inverse functions gives $\arcsin'(x) = \dfrac{1}{\sin'(y)} = \dfrac{1}{\cos(y)}$. Since $-\pi/2 \le y \le \pi/2$, $\cos(y) \ge 0$, so $\cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-x^2}$ and $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

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4. a) Substitute $u = 1+\sqrt{x}$, $du = \dfrac{dx}{2\sqrt{x}}$, or $dx = 2\sqrt{x}\,du = 2(u-1)\,du$, to get $\displaystyle\int \frac{dx}{1+\sqrt{x}} = \int \frac{2(u-1)}{u}\,du = 2\int 1 - \frac{1}{u}\,du = 2u - 2\log(u) + C = 2(1+\sqrt{x}) - 2\log(1+\sqrt{x}) + C$.

b) Integrate by parts with $u = x$, $du = dx$, $dv = e^{-x}dx$, $v = -e^{-x}$, to get
$\displaystyle\int_0^\infty xe^{-x}\,dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx = 0 - 0 - e^{-x}\Big|_0^\infty = 0 + e^0 = 1$

c) The partial fraction decomposition has the form $\dfrac{x+1}{x^2(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1}$. Solve for the constants to get $A = B = 1$ and $C = D = -1$. Therefore,
$\displaystyle\int \frac{x+1}{x^4+x^2}\,dx = \int \frac{1}{x} + \frac{1}{x^2} - \frac{x}{x^2+1} - \frac{1}{x^2+1}\,dx = \log(x) - \frac{1}{x} - \frac{1}{2}\log(x^2+1) - \arctan(x) + C.$

d) Substitute $u = \tan(x/2)$, $dx = \dfrac{2du}{1+u^2}$, $\sin(x) = \dfrac{1-u^2}{1+u^2}$, $\cos(x) = \dfrac{2u}{1+u^2}$, to get $\displaystyle\int \frac{dx}{1+\sin(x)+\cos(x)} = \int \frac{\frac{2}{1+u^2}\,du}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} = \int \frac{2du}{1+u^2+2u+1-u^2} = \int \frac{du}{1+u} = \log(1+u) + C = \log(1+\tan(x/2)) + C.$

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5. a) $\displaystyle\lim_{x\to 0} \frac{\cos(2x^3) - \exp(-2x^6)}{x^6\log(1+x^6)} = \lim_{x\to 0} \frac{1 - \frac{1}{2}(2x^3)^2 + \frac{1}{4!}(2x^3)^4 + o(x^{17}) - (1 - 2x^6 + \frac{1}{2}(2x^6)^2 + o(x^{17}))}{x^6(x^6 + o(x^1 1))}$
$\displaystyle= \lim_{x\to 0} \frac{(\frac{2}{3} - 2)x^{12} + o(x^{17})}{x^{12} + o(x^{17})} = \frac{2}{3} - 2 = -\frac{4}{3}$

b) Substitute $y = 1/x$ to get $\displaystyle\lim_{x\to\infty}\Big(x^2 - x^3\sin\frac{1}{x}\Big) = \lim_{y\to 0^+} \frac{y - \sin(y)}{y^3} = \lim_{y\to 0^+} \frac{y - (y - \frac{1}{3!}y^3 + o(y^4))}{y^3}$
$\displaystyle= \lim_{y\to 0^+} \frac{1}{3!} + o(y) = \frac{1}{6}$ or use L'Hôpital's Rule, $\displaystyle\lim_{y\to 0^+} \frac{y - \sin(y)}{y^3} = \lim_{y\to 0^+} \frac{1-\cos(y)}{3y^2} = \lim_{y\to 0^+} \frac{\sin(y)}{6y} = \frac{1}{6}$.

c) $\displaystyle\lim_{x\to 1} \frac{3x^4 - 8x^3 + 12x - 7}{x^4 - 4x + 3} = \lim_{x\to 1} \frac{12x^3 - 24x^2 + 12}{4x^3 - 4} = \lim_{x\to 1} \frac{36x^2 - 48x}{12x^2} = \frac{36-48}{12} = -1$ [L'Hôpital's Rule]

d) $\displaystyle\lim_{n\to\infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}} = \lim_{n\to\infty} \frac{1 + (-1/2)^n}{2 - (-1/2)^n} = \frac{1}{2}$, since $\displaystyle\lim_{n\to\infty}(-1/2)^n = 0$.

6. a) $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n^2+3n}=\sum_{n=1}^{\infty}\Big(\frac{1}{n}-\frac{1}{n+3}\Big)=\sum_{n=1}^{\infty}\Big\{\Big(\frac{1}{n}-\frac{1}{n+1}\Big)+\Big(\frac{1}{n+1}-\frac{1}{n+2}\Big)+\Big(\frac{1}{n+2}-\frac{1}{n+3}\Big)\Big\}$
$\displaystyle=\Big(\frac{1}{1}-0\Big)+\Big(\frac{1}{1+1}-0\Big)+\Big(\frac{1}{1+2}-0\Big)=\frac{11}{6}$ [three telescoping sums]

b) $\displaystyle\sum_{n=1}^{\infty}\frac{a^n+(-1)^{n+1}}{(a+1)^{2n}}=\sum_{n=1}^{\infty}\Big(\frac{a}{(a+1)^2}\Big)^n-\sum_{n=1}^{\infty}\Big(\frac{-1}{(a+1)^2}\Big)^n=\frac{a/(a+1)^2}{1-a/(a+1)^2}+\frac{1/(a+1)^2}{1+1/(a+1)^2}$

c) $\displaystyle\sum_{n=1}^{\infty}n^2x^n=x\frac{d}{dx}\sum_{n=0}^{\infty}nx^n=x\frac{d}{dx}\Big(x\frac{d}{dx}\sum_{n=0}^{\infty}x^n\Big)=x\frac{d}{dx}\Big(x\frac{d}{dx}\frac{1}{1-x}\Big)=x\frac{d}{dx}\Big(\frac{x}{(1-x)^2}\Big)=\frac{x(1+x)}{(1-x)^3}$

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7. a) $\displaystyle\sum_{n=1}^{\infty}\Big(\frac{1-n}{1+n}\Big)^n$ diverges since $\displaystyle\lim_{n\to\infty}\Big|\frac{1-n}{1+n}\Big|^n=\lim_{n\to\infty}\frac{(1-1/n)^n}{(1+1/n)^n}=\frac{e^{-1}}{e}\neq 0.$

b) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n^2-1}}$ is alternating and converges by Leibniz's Rule since the terms decrease to 0. The convergence is conditional since $\displaystyle\lim_{n\to\infty}\frac{1/n}{1/\sqrt{n^2-1}}=1$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2-1}}$ also diverges by LCT.

c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}2^{\sqrt{n}}}$ converges absolutely by the Integral Test: $\displaystyle\int_2^{\infty}\frac{1}{\sqrt{x}2^{\sqrt{x}}}\,dx=2\int_{\sqrt{2}}^{\infty}2^{-u}\,du=-\frac{2^{1-u}}{\log(2)}\Big|_2^{\infty}<\infty.$

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8. a) $\displaystyle 1+\frac{1}{2}x+\frac{1}{4\cdot 3}x^2+\frac{1}{6\cdot 5\cdot 4}x^3+\frac{1}{8\cdot 7\cdot 6\cdot 5}x^4+\cdots=\sum_{n=0}^{\infty}\frac{n!}{(2n)!}x^n$ converges for all x, since
$\displaystyle\lim_{n\to\infty}\frac{(n+1)!}{(2n+2)!}|x|^{n+1}\Big/\frac{n!}{(2n)!}|x|^n=\lim_{n\to\infty}\frac{(n+1)}{(2n+2)(2n+1)}|x|=0$ for all x.

b) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n^2}x^{2n}$. $\displaystyle\lim_{n\to\infty}\frac{3^{n+1}}{(n+1)^2}|x|^{2n+2}\Big/\frac{3^n}{n^2}|x|^{2n}=\lim_{n\to\infty}3\Big(\frac{n}{n+1}\Big)^2|x|^2=3|x|^2$, so $r=1/\sqrt{3}$ by the Ratio Test. At the endpoints, $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n^2}\Big(\frac{\pm 1}{\sqrt{3}}\Big)^{2n}=\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$ by the Integral Test, so the interval is $[-1/\sqrt{3},1/\sqrt{3}]$.

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9. a) $\displaystyle\int_0^x e^{-t^2}\,dt=\int_0^x\sum_{n=0}^{\infty}\frac{1}{n!}(-t^2)^n\,dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}x^{2n+1}$ [termwise integration]

b) $\displaystyle\int_0^1 e^{-t^2}\,dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}$. Since the sum is alternating, $\displaystyle|s-s_n|<|a_{n+1}|=\frac{1}{(2n+3)(n+1)!}<10^{-4}\Leftrightarrow$
$(2n+3)(n+1)!>10^4\Leftrightarrow n\geq 6$. Therefore, the integral is approximated by
$\displaystyle 1-\frac{1}{3}+\frac{1}{5\cdot 2}-\frac{1}{7\cdot 3!}+\frac{1}{9\cdot 4!}-\frac{1}{11\cdot 5!}+\frac{1}{13\cdot 6!}=0.746836$ with an error $\displaystyle<\frac{1}{15\cdot 7!}=1.32\times 10^{-5}$.
