ABEL'S THEOREM

MATH 166: CALCULUS II

Theorem (Abel)**.** *Suppose* $\sum_{n=0}^{\infty} a_n$ *converges. Then*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges absolutely for $|x| < 1$ *and*

$$\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$$

Proof. Since $a_n \to 0$, there is a positive integer N such that $|a_n| < 1$ for all $n \geq N$. Then $|a_n x^n| = |a_n||x|^n < |x|^n$ for all $n \geq N$. The series $\sum_{n=0}^{\infty} |x|^n$ converges for $|x| < 1$, so the series $\sum_{n=0}^{\infty} |a_n x^n|$ converges by the Comparison Theorem.

To prove the statement about the limit, we must show that given $\epsilon > 0$ there is a $\delta > 0$ such that if $1 - \delta < x < 1$ then $|f(x) - s| < \epsilon$ where $s = \sum_{n=1}^{\infty} a_n$. The usual strategy is to try to write $f(x) - s = (1 - x) \cdot (\text{something})$ so that as $x \to 1$, $f(x) \to s$. Abel found a clever way to do this with series: Let $s_k = \sum_{n=0}^{k} a_n$ and define $s_{-1} = 0$, so that $a_n = s_n - s_{n-1}$ for all $n \geq 0$. Then,

$$\begin{aligned}
\sum_{n=0}^{k} a_n x^n &= \sum_{n=0}^{k} (s_n - s_{n-1}) x^n = \sum_{n=0}^{k} s_n x^n - \sum_{n=1}^{k} s_{n-1} x^n \\
&= \sum_{n=0}^{k} s_n x^n - \sum_{n=0}^{k-1} s_n x^{n+1} \quad \text{[shift index]} \\
&= s_k x^k + \sum_{n=0}^{k-1} s_n x^n (1 - x)
\end{aligned}$$

Now assume $|x| < 1$ and let $k \to \infty$. Since $s_k x^k \to s \cdot 0 = 0$, we find

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = (1 - x) \sum_{n=0}^{\infty} s_n x^n$$

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The formula for a geometric series implies that $1 = (1-x)\sum_{n=0}^{\infty} x^n$,

so $s = (1-x)\sum_{n=0}^{\infty} sx^n$ and

$$f(x) - s = (1-x)\sum_{n=0}^{\infty} s_n x^n - (1-x)\sum_{n=0}^{\infty} sx^n = (1-x)\sum_{n=0}^{\infty}(s_n - s)x^n$$

Given $\epsilon > 0$, there is a positive integer N such that $|s_n - s| < \frac{\epsilon}{2}$ whenever $n \geq N$. Therefore, assuming $0 < x < 1$,

$$\begin{aligned}
|f(x) - s| &= (1-x)\Big|\sum_{n=0}^{\infty}(s_n - s)x^n\Big| \leq (1-x)\sum_{n=0}^{\infty}|s_n - s|x^n \\
&= (1-x)\sum_{n=0}^{N}|s_n - s|x^n + (1-x)\sum_{n=N}^{\infty}|s_n - s|x^n \\
&< (1-x)\sum_{n=0}^{N}|s_n - s| + (1-x)\sum_{n=N}^{\infty}\frac{\epsilon}{2}x^n \\
&< (1-x)K + \frac{\epsilon}{2}
\end{aligned}$$

where $K = \sum_{n=0}^{N}|s_n - s|$. The last inequality follows by adding the geometric series: $(1-x)\sum_{n=N}^{\infty}\frac{\epsilon}{2}x^n = \frac{\epsilon}{2}x^N < \frac{\epsilon}{2}$. We can make $(1-x)K$ small too: $(1-x)K < \frac{\epsilon}{2}$ if and only if $1 - \frac{\epsilon}{2K} < x$. Therefore, if we take $\delta = \frac{\epsilon}{2K}$, then $1 - \delta < x < 1$ implies that

$$|f(x) - s| < (1-x)K + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which shows $\lim_{x\to 1^-} f(x) = s$. □

Example. Integrating the geometric series

$$\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}, \qquad |x| < 1$$

gives

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}x^n = \log(1+x), \qquad |x| < 1$$

The series also converges for $x = 1$, since it is alternating and the terms decrease to 0. By Abel's Theorem the value of this series is $\lim_{x\to 1^-}\log(1+x) = \log(2)$:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log(2)$$