DEPARTMENT OF MATHEMATICS

SOLUTIONS FOR THE AUMAN PRIZE
2000

(1) Let $f(x)$ be differentiable at $x = a$, and $f(a) \neq 0$.

$$\begin{aligned}
\lim_{n\to\infty}\left[\frac{f(a+1/n)}{f(a)}\right]^n &= \lim_{x\to 0^+}\left[\frac{f(a+x)}{f(a)}\right]^{1/x} \\
&= \lim_{x\to 0^+}\exp\big([\log f(a+x) - \log f(a)]/x\big) \\
&= \exp\big(\lim_{x\to 0^+}[\log f(a+x) - \log f(a)]/x\big) \\
&= \exp\big(\lim_{x\to 0^+}[f'(a+x)/f(a+x)\big) \quad \text{[L'Hôpital]} \\
&= \exp\big(f'(a)/f(a)\big)
\end{aligned}$$

(2) Prove that there is no continuous function f with domain the closed interval $[0, 1]$ and range the open interval $(0, 1)$.

Proof. Assume f is continuous on $[0, 1]$. By the Extreme Value Theorem there exist $a, b \in [0, 1]$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in [0, 1]$. By the Intermediate Value Theorem, the range of f must include all numbers between $f(a)$ and $f(b)$ and therefore equals the closed interval $[f(a), f(b)]$. No closed interval can equal an open interval. If $[f(a), f(b)] = (0, 1)$, for example, we would have $0 < f(a) < 1$. But then $0 < f(a)/2 < f(a)$, so $f(a)/2 \in (0, 1)$ but $f(a)/2 \notin [f(a), f(b)]$, a contradiction. Therefore, the range of f cannot be $(0, 1)$.

(3)

$$\begin{aligned}
\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}\,dt &= \lim_{x\to\infty} xe^{-x^2}\int_0^x e^{t^2}\,dt \\
&= \lim_{x\to\infty}\frac{\int_0^x e^{t^2}\,dt}{e^{x^2}/x} \\
&= \lim_{x\to\infty}\frac{e^{x^2}}{2e^{x^2} - e^{x^2}/x^2} \quad \text{[L'Hôpital]} \\
&= \lim_{x\to\infty}\frac{1}{2-1/x^2} = \frac{1}{2}
\end{aligned}$$

(4) Determine how far a stack of n identical books can be made to project over the edge of a table by stacking them one on top of the other. Determine the theoretical limit of this overhang as $n \to \infty$.

Solution. Let's take the length of the book to be one unit and label the books b_1 through b_n from the top down. In order to stack the books so that each book extends as far as possible from the edge of the book below it without falling, we must arrange for the center of gravity, c_k, of books b_1 through b_{k-1} together to lie at the right edge of book b_k. Calculating distances from the left edge of b_k, we see that c_k is the weighted average of the center of gravity of b_k—a distance of $1/2$ and weight 1—and the center of gravity of b_1 through b_{k-1} together—a distance of 1 (the right edge of b_k) and weight $k-1$:

$$c_k = \frac{1 \cdot \frac{1}{2} + (k-1) \cdot 1}{k} = 1 - \frac{1}{2k}$$

This means that the distance b_k projects from the book below it is $1 - c_k = 1/(2k)$. Therefore, the total distance the books project over the edge of the table together is:

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} = \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \approx \frac{1}{2}\log(n)$$

In fact, comparing areas of rectangles above and below the graph of $1/x$, we find that

$$\log(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < \log(n) + 1$$

Since $\log(n) \to \infty$ as $n \to \infty$, there is no *theoretical* limit to the overhang, although there are obvious practical limitations.

(5) Show if there are $n \geq 2$ people in a room, then two of them know the same number of people among those present. (Assume that if A knows B, then B knows A).

Solution. If the statement is false there must be a group of n people, each of whom knows a different number of people in the group. Since any list of n distinct integers, each between 0 and $n-1$, must contain all of the numbers from 0 through $n-1$ exactly once, it follows that there must be one person in the group who knows nobody, and one person who knows everybody. But then the person who knows nobody would know the person who knows everybody, a contradiction. Therefore, no such group of people exists.