DIFFERENTIATING UNDER THE INTEGRAL

MATH 166: HONORS CALCULUS II

There is a certain technique for evaluating integrals that is no longer taught in the standard calculus curriculum. It is mentioned in the autobiography of the renowned physicist Richard Feynman, *Surely You're Joking Mr. Feynman*, p.86–87:

> So every physics class, I paid no attention to what was going on with Pascal's Law, or whatever they were doing. I was up in the back with this book: *Advanced Calculus*, by Woods. Bader [Feynman's High School Physics teacher, who loaned Feynman his copy of Wood's book] knew I had studied *Calculus for the Practical Man* a little bit, so he gave me the real works—it was for a junior or senior course in college. It had Fourier series, Bessel functions, determinants, elliptic functions—all kinds of wonderful stuff that I didn't know anything about.
>
> That book also showed how to differentiate parameters under the integral sign—it's a certain operation. It turns out that it's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method, and I used that one damn tool again and again. So because I was self-taught using that book, I had peculiar methods of doing integrals.
>
> The result was, when guys at MIT or Princeton had trouble doing a certain integral, it was because they couldn't do it with the standard methods they had learned in school. If it was contour integration, they would have found it; if it was a simple series expansion, they would have found it. Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me.

The method is illustrated by the following example. To evaluate

$$\int_0^1 \frac{x^5 - 1}{\log(x)}\, dx$$

first introduce a parameter, say α, into the integral, thereby creating a function of this parameter.

$$f(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log(x)}\, dx$$

The problem now is to find $f(5)$. This will be done by computing $f'(\alpha)$—taking the derivative with respect to α inside the integral—then integrating with respect

Date: Spring, 2000.

to x to get an explicit expression for $f'(\alpha)$. Finally, integrating this expression with respect to α gives an expression for $f(\alpha)$:

$$f'(\alpha) = \int_0^1 \frac{d}{d\alpha}\Big(\frac{x^\alpha - 1}{\log(x)}\Big)\, dx = \int_0^1 \frac{x^\alpha \log(x)}{\log(x)}\, dx = \int_0^1 x^\alpha\, dx = \frac{x^{\alpha+1}}{\alpha+1}\Big|_0^1 = \frac{1}{\alpha+1}$$

Now integrating with respect to α gives

$$f(\alpha) = \int \frac{1}{\alpha+1}\, d\alpha = \log(\alpha+1) + c$$

Since $f(0) = 0$, we see that $c = 0$ and $f(\alpha) = \log(\alpha+1)$. The answer to our original problem is then $f(5) = \log(6)$. In fact, we have derived the general formula

$$\int_0^1 \frac{x^\alpha - 1}{\log(x)}\, dx = \log(\alpha+1)$$

The method of "differentiating under the integral" appears in many old calculus texts, mainly those published before 1945. See, for example, Edwin Wilson's text, *Advanced Calculus*, published in 1912, where examples and a rigorous justification for the method are given on p.281–288.

Here are a few other definite integrals that can be evaluated in this manner.

$$\int_0^\pi \log(1 + \alpha\cos(x))\, dx = \pi \log \frac{1 + \sqrt{1-\alpha^2}}{2}$$

$$\int_0^\pi \log(1 - 2u\cos(x) + \alpha^2)\, dx = \begin{cases} \pi \log \alpha^2 & \alpha^2 \geq 1 \\ 0 & \alpha^2 \leq 1 \end{cases}$$

$$\int_0^{\pi/2} \frac{\log(1 + \cos(\alpha)\cos(x))}{\cos(x)}\, dx = \frac{1}{2}\Big(\frac{\pi^2}{4} - \alpha^2\Big)$$

$$\int_0^1 x^\alpha (\log(x))^n\, dx = (-1)^n \frac{n!}{(\alpha+1)^{n+1}}$$

$$\int_0^\infty \frac{dx}{(x^2+\alpha^2)^{n+1}} = \frac{\pi}{2}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n\cdot \alpha^{2n+1}}$$

Keep in mind that to successfully apply this method some experimentation may be needed to discover an appropriate way to introduce a parameter into the integral. So, for example, to evaluate

$$\int_0^{\pi/2} \frac{\log(1 + (1/2)\cos(x))}{\cos(x)}\, dx$$

it may take some time to realize that replacing $1/2$ by $\cos(\alpha)$ leads to a solution. The answer, according to the above formula with $\alpha = \frac{\pi}{3}$, is $\frac{1}{2}\Big(\frac{\pi^2}{4} - \frac{\pi^2}{9}\Big) = \frac{5\pi^2}{72}$.