INFINITE EXPONENTIATION

MATH 166: HONORS CALCULUS II

We can make sense of an infinite exponential, $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, using a recursive definition:

$$f_1(x) = x, \qquad f_n(x) = x^{f_{n-1}(x)}$$

The infinite exponential function $f(x)$ can then be defined to be the limit of $f_n(x)$ as n goes to infinity. This limit will not exist for every x. For example, $f_n(1) = 1$ for all n so $f(1) = 1$, but $f_n(2)$ would quickly go to infinity as n increases. To see which values might work, we'll first plot and compare a few of the functions $f_n(x)$.

There is some unusual behavior near 0 and the graphs split apart around $x = 1.4$, but the different functions $f_n(x)$ agree closely between these numbers. The key to understanding $f(x)$ is to realize that if this limit exists then the recursive definition $f_n(x) = x^{f_{n-1}(x)}$ implies $f(x) = x^{f(x)}$. Let $y = f(x)$. Then $y = x^y$ and we can solve: $x = y^{1/y}$. The function $y^{1/y}$ is the inverse of $f(x)$, and vice-versa, at least on intervals containing 1 where these functions are 1–1. We can easily check for such intervals for the function $y^{1/y}$.

Since $y^{1/y} = e^{\log(y)/y}$, we compute the derivative to be

$$\frac{d}{dy}[y^{1/y}] = y^{1/y}\Big[-\frac{\log(y)}{y^2} + \frac{1}{y^2}\Big] = y^{1/y-2}(1 - \log(y))$$

which confirms that $y^{1/y}$ is strictly increasing for $1 - \log(y) \geq 0$, i.e., $0 < y \leq e$ (this interval contains 1) and strictly decreasing for $y > e$. Therefore the desired inverse $f(x)$ exists for $0 < x < e^{1/e} = 1.44467$.

The plots confirm that the graphs of $f_n(x)$ split apart near $x = 1.4$:

The graphs of $f_n(x)$ seem to split apart near 0. When n is odd, $f_n(x)$ goes to 0 and when n is even $f_n(x)$ goes to 1. However, as n increases, these functions do approach a common limit: the "even" functions shrink down and the "odd" functions creep up.

We can obtain the graph of the inverse of $y^{1/y}$ by plotting it parametrically (in *Mathematica*, `ParametricPlot[{y^(1/y),y},{y,0,E}]`). This plot agrees closely to the plots of $f_n(x)$ above.

This analysis proves that $f(x)$ exists for $0 < x < e^{1/e}$. More work would have to be done to prove that the limit of $f_n(x)$ as n goes to infinity does not exist when $x > e^{1/e}$. Just because an inverse to $y^{1/y}$ cannot be defined for $y > e$ does not mean that the limit does not exist. The graphs do indicate, however, that $f_n(x)$ goes to infinity as n goes to infinity if $x > e^{1/e}$.