A FORMULA FOR PASCAL'S TRIANGLE

MATH 166: HONORS CALCULUS II

The sum of the numbers on a diagonal of Pascal's triangle equals the number below the last summand. For example, $1+2=3$, $1+2+3=6$, $1+3=4$, $1+3+6=10$, etc.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\cdots & & & & & & & & & &
\end{array}
$$

This fact is expressed formally in the identity:

$$\sum_{k=0}^{p} \binom{k+n-1}{n-1} = \binom{p+n}{n}$$

Here is one simple way to prove the identity. First observe that $\binom{k+n-1}{n-1}$ is the number of ways of dividing k objects into n subsets: line up $k+n-1$ objects and select $n-1$ of them to mark the boundaries of the n subsets. The number of ways of choosing $n-1$ from $k+n-1$ is, of course, $\binom{k+n-1}{n-1}$. Now the sum on the left hand side is the number of ways of dividing less than or equal to p objects into n subsets, one term for each number of objects $k=0,\ldots,p$. The right hand side is the number of ways of dividing p objects into $n+1$ subsets. By ignoring the first subest, every way of dividing p objects into n subsets gives exactly one way of dividing k objects into $n-1$ subsets where k is p minus the number of elements in the ignored subset. Conversely, every way of dividing $k \leq p$ objects into n subsets gives rise to exactly one way of dividing p objects into $n+1$ subsets: just add a subset with $p-k$ objects. Therefore, the two sides of the above identity must be equal.

Date: Spring 2000.