CONSTRUCTION OF THE PENTAGON

MATH 166: HONORS CALCULUS II

(1) Construct a circle and a diameter AB.
(2) Construct a another diameter perpendicular to AB which meets the circle at D. (Find the point C by marking an arc centered at A with radius AB and then doing the same with B as the center.)
(3) Bisect the segment DO at the point G. (Find the points E and F on the circle by marking arcs centered at D with radius DO.)
(4) Construct an arc centered at G with radius GB so that it meets the perpendicular diameter at a point H.
(5) Construct an arc centered at A with radius AH so that it meets the circle at the point I. Mark off the point J with an arc centered at A with radius AI. Mark off the points K and L with arcs centered at I and J, respectively, and radius AI. The points A,I, J, K, and L are the vertices of a regular pentagon.

Proof. Let us assume for simplicity that the radius of the circle is 1. (For a circle of radius R, all lengths below would be scaled by a factor of R.) We first calculate some lengths in the diagram. From the right triangle GOB, we compute that $|GB| = \sqrt{(1/2)^2 + 1} = \sqrt{5}/2$. Thus, $|DH| = 1/2 + \sqrt{5}/2 = \phi$, the golden mean, and $|OH| = |DH| - 1 = -1/2 + \sqrt{5}/2$. From the right triangle AOH, $|AH| = \sqrt{1 + (-1/2 + \sqrt{5}/2)^2} = \sqrt{(5 - \sqrt{5})/2}$. Let $AIJKL$ be the pentagon inscribed in the circle as indicated in the diagram. The proof will be complete if we show that $|AH| = |AI|$.

Let $s = |AI|$. Consider the isosceles triangle AOI. Since the pentagon in comprised of 5 of these triangles all sharing the vertex O, the angle AOI is clearly $2\pi/5$. Bisecting this angle produces two equal right triangles in AOI and we conclude $s/2 = \sin(\pi/5)$. Applying the formula $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$ repeatedly, we arrive at the equation

$$\sin(5\theta) = 16\sin^5(\theta) - 20\sin^3(\theta) + 5\sin(\theta)$$

Letting $\theta = \pi/5$ we obtain

$$0 = 16(s/2)^5 - 20(s/2)^3 + 5(s/2) = (s/2)(s^4 - 5s^2 + 5)$$

Since $0 < \sin(\pi/5) < \sin(\pi/4) = \sqrt{2}/2$, we are forced to conclude using the quadratic formula that $s^2 = (5 - \sqrt{5})/2$, and $s = \sqrt{(5 - \sqrt{5})/2}$. Therefore, $|AH| = s$ as claimed.

□

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