INTEGRABILITY ON SUBINTERVALS

MATH 166: CALCULUS II

The purpose of this note is to fill in a gap in Apostol's Calculus text. The following theorem is very useful in practice and simplifies the statements of several theorems in the text.

Theorem. *Let f be a function defined on an interval $[a,b]$. If f is integrable on $[a,b]$, then f is integrable on any subinterval $[c,d] \subset [a,b]$.*

Proof. We shall prove the logically equivalent statement: Suppose there exists a subinterval $[c,d] \subset [a,b]$ on which f is not integrable. Then f is not integrable on the larger interval $[a,b]$.

To show f is not integrable on $[a,b]$ we must show that there is a gap between the upper and lower integrals of f. In other words, there is a constant M such that given any step functions s and t on $[a,b]$ satisfying $s(x) \leq f(x) \leq t(x)$ for $a \leq x \leq b$, we have

$$\int_a^b s - \int_a^b t \geq M$$

Since we are assuming f is not integrable on $[c,d]$, we know there is a constant M such that

$$\int_c^d s - \int_c^d t \geq M$$

This inequality immediately implies the previous one by additivity of intervals and linearity of integrals for step functions:

$$\int_a^b s - \int_a^b t = \int_a^c (s-t) + \int_c^d (s-t) + \int_d^b (s-t) \geq 0 + M + 0$$

□