SUBSTITUTIONS IN LIMITS

MATH 166: CALCULUS II

In this note we examine how to make a "change of variables" or a "substitution" when calculating limits. In the following theorem, we assume that the substitution function $u = g(x)$ is continuous and non-constant, but we do not to assume anything about the function it is composed with.

Theorem. *Let $g(x)$ be a function defined on an interval $[a, b]$. Assume that $g(x)$ is continuous and non-constant on any subinterval of $[a, b]$ containing p. Define p^* to be the symbol p^+ if $p = a$, or p^- if $p = b$, or p otherwise. Similarly, let $q = g(p)$ and define q^* to be the symbol q^+ if $g(x)$ has a relative minimum at p, or q^- if $g(x)$ is a relative maximum at p, or q otherwise. Then for any function $f(u)$ defined in an appropriate neighborhood of q we have*

$$\lim_{x \to p*} f(g(x)) = \lim_{u \to q^*} f(u)$$

Proof. We must show the existence of one limit implies the existence of the other and that the two limiting values are equal.

Let us first assume $\lim_{x \to p^*} f(g(x)) = L$ exists and prove $\lim_{u \to q^*} f(u) = L$. This part is similar to the proof of Theorem 3.10, Apostol p.147. By the definition of limit, given $\epsilon > 0$, there exists an interval I containing p such that if $x \in I$, then $|f(g(x)) - L| < \epsilon$. If $g(x)$ has a relative extremum at $x = p$, we replace I by a smaller interval containing p so that so that $q = g(p)$ is an extreme value of $g(x)$ restricted to I. Let J be a closed subinterval of I containing p. Since $g(x)$ is continuous and non-constant on any subinterval containing p, the Extreme Value Theorem and the Intermediate Value Theorem imply that the range of $g(x)$ restricted to J is also a closed interval, say $K = [c, d]$. Note that if $g(x)$ has a relative minimum at p then $q = c$, or if $g(x)$ has a relative maximum at p then $q = d$. By construction, for any $u \in K$, there is an $x \in J \subset I$ such that $u = g(x)$. Then, $|f(u) - L| = |f(g(x)) - L| < \epsilon$, since $x \in I$. This proves $\lim_{u \to q^*} f(u) = L$.

Let us now assume $\lim_{u \to q^*} f(u) = L$ exists and prove $\lim_{x \to p^*} f(g(x)) = L$. This part is similar to the proof of Theorem 3.5, Apostol p.141. By the definition of limit, given $\epsilon > 0$, there exists an interval K containing q such that if $u \in K$ then $|f(u) - L| < \epsilon$. Since $g(x)$ is continuous at p (and this is all we need to assume about $g(x)$ for this part of the proof), there is an interval I containing p such that if $x \in I$ then $g(x) \in K$, which implies $|f(g(x)) - L| < \epsilon$. This proves $\lim_{x \to p^*} f(g(x)) = L$. $\square$

Example. $\lim_{x \to 0} \frac{\sin(x^2)}{x^2} = \lim_{u \to 0^+} \frac{\sin(u)}{u} = 1$

Date: Spring 2000.

Example. To see why the assumptions of the theorem are necessary, consider the function

$$g(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{1}{n+1} < |x| \leq \frac{1}{n} \text{ for some } n \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases}$$

Then $g(x)$ is non-constant in any interval containing 0 and is also continuous at 0 since $\lim_{x\to 0} g(x) = g(0) = 0$. However, $g(x)$ is not continuous in any *interval* containing 0 since for $n \in \mathbf{N}$,

$$\lim_{x\to \frac{1}{n}^-} g(x) = \frac{1}{n} \neq \frac{1}{n-1} = \lim_{x\to \frac{1}{n}^+} g(x)$$

If we define

$$f(u) = \begin{cases} 1 & \text{if } |u| = \frac{1}{n} \text{ for some } n \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases}$$

then clearly $f(g(x)) = 1$ for all $0 < |x| \leq 1$ so $\lim_{x\to 0} f(g(x)) = 1$. Yet $\lim_{u\to 0^+} f(u)$ does not exist since $f(u)$ takes on the values 0 and 1 for arbitrarily small positive values of u.