o-SYMBOLS

MATH 166: HONORS CALCULUS II

Definitions

The symbol $o(g(x))$ stands for a function $f(x)$ that satisfies

$$\lim_{x\to a}\frac{f(x)}{g(x)}=0.$$

The limit is usually taken as $x \to 0$. When it is some other number, $a \neq 0$, the number a will usually be mentioned explicitly. Our primary use of o-symbols is with powers of x. So, for example, we use the term $o(x^3)$ to stand for an arbitrary function $f(x)$ that satisfies $f(x)/x^3 \to 0$ as $x \to 0$. These o-symbols are often very useful for calculating Taylor polynomials and finding limits.

As an extension of the notation, we write "$f(x) = h(x) + o(g(x))$" to mean that

$$\lim_{x\to a}\frac{f(x)-h(x)}{g(x)}=0.$$

For example, the Lagrange form of the remainder term in Taylor's formula is

$$E_nf(x;a)=\frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1}.$$

Since

$$\lim_{x\to a}\frac{E_nf(x;a)}{(x-a)^n}=\lim_{x\to a}\frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)=\frac{f^{(n+1)}(a)}{(n+1)!}\cdot 0=0$$

we see that $E_nf(x;a)=o\big((x-a)^n\big)$ and we may express Taylor's formula as

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)(a)}}{k!}(x-a)^k+o\big((x-a)^n\big).$$

This allows us to rewrite any expression that involves complicated functions in terms of simple polynomials and o-terms. We already know how to combine and simplify polynomials. To be able to simplify the more general expressions we need to know how to combine and simplify o-terms.

Algebra of o-Symbols

Theorem 1. *As $x \to a$ we have,*

(1) $o(g(x)) \pm o(g(x)) = o(g(x))$
(2) $o(cg(x)) = o(g(x)) \qquad c \neq 0$
(3) $f(x)\cdot o(g(x)) = o(f(x))o(g(x)) = o(f(x)g(x))$
(4) $o(o(g(x))) = o(g(x))$
(5) $\dfrac{1}{1-g(x)} = 1 + g(x) + o(g(x))$ *if $g(x) \to 0$ as $x \to a$.*

Proof. 1. The symbol $o(g(x))\pm o(g(x))$ stands for a function of the form $f(x)\pm h(x)$ where we know nothing about $f(x)$ or $h(x)$ other than the limits: $f(x)/g(x)\to 0$ and $h(x)/g(x)\to 0$ as $x\to a$. That is enough information to compute the limit

$$\lim_{x\to a}\frac{f(x)\pm h(x)}{g(x)}=\lim_{x\to a}\frac{f(x)}{g(x)}\pm\lim_{x\to a}\frac{h(x)}{g(x)}=0\pm 0=0.$$

This shows that $f(x)\pm h(x)=o(g(x))$. Therefore, a function of the form $o(g(x))\pm o(g(x))$ is also of the form $o(g(x))$ and we write this fact as $o(g(x))\pm o(g(x))=o(g(x))$.

2. The symbol $o(cg(x))$ stands for a arbitrary function $f(x)$ that has the property $f(x)/(cg(x))\to 0$ as $x\to a$. But then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=c\cdot\lim_{x\to a}\frac{f(x)}{cg(x)}=c\cdot 0=0$$

Therefore $f(x)$ is also of type $o(g(x))$.

3. The symbol $f(x)o(g(x))$ stands for a function of the form $f(x)h(x)$ where we know nothing about $f(x)$ and the only thing we know about $h(x)$ is that $h(x)/g(x)\to 0$ as $x\to a$. This is enough information to conclude that

$$\lim_{x\to a}\frac{f(x)h(x)}{f(x)g(x)}=\lim_{x\to a}\frac{h(x)}{g(x)}=0.$$

By definition, this implies that $f(x)h(x)=o(f(x)g(x))$ and that is what is meant by "$f(x)o(g(x))=o(f(x)g(x))$." Similarly, $o(f(x))o(g(x))$ stands for a function of the form $k(x)h(x)$ where $k(x)/f(x)\to 0$ and $h(x)/g(x)\to 0$ as $x\to a$. As above, it is easy to see that $k(x)h(x)/(f(x)g(x))\to 0$ so $k(x)h(x)$ is of type $o(f(x)g(x))$ as well and "$o(f(x))o(g(x))=o(f(x)g(x))$."

4. The symbol $o(o(g(x)))$ is a little harder to decode. Starting on the inside, $o(g(x))$ stands for an arbitrary function $h(x)$ satisfying $h(x)/g(x)\to 0$ as $x\to a$. So $o(o(g(x)))$ is really $o(h(x))$ and stands for an arbitrary function $f(x)$ satisfying $f(x)/h(x)\to 0$ as $x\to a$. Putting these two limits together we get:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)}{h(x)}\frac{h(x)}{g(x)}=\lim_{x\to a}\frac{f(x)}{h(x)}\lim_{x\to a}\frac{h(x)}{g(x)}=0\cdot 0=0.$$

Therefore, $f(x)$ is also of type $o(g(x))$ and that is what is meant by "$o(o(g(x)))=o(g(x))$."

5. Simple algebra implies that for any number $u\neq 1$ we have

$$\frac{1}{1-u}=1+u+\frac{u^2}{1-u}.$$

Since $\left.\frac{u^2}{1-u}\right/u=\frac{u}{1-u}\to 0$ as $u\to 0$ we may write this as

$$\frac{1}{1-u}=1+u+o(u).$$

We are assuming that $g(x)\to 0$ as $x\to a$, so we may replace u with $g(x)$ to get

$$\frac{1}{1-g(x)}=1+g(x)+o(g(x)).$$

□

We will usually apply the above rules to o-terms of the form $o(x^n)$ or $o\big((x-a)^n\big)$, so it is helpful to see what the above statements imply when $f(x) = x^n$ and $g(x) = x^m$. Modifications for powers of $(x-a)$ should be obvious.

Corollary 2.

(1) $x^n = o(x^m)$ *for* $m < n$
(2) *If* $f(x) = o(x^n)$ *then* $f(x) = o(x^m)$ *for* $m \leq n$.
(3) $x^n o(x^m) = o(x^n)o(x^m) = o(x^{n+m})$
(4) $o(x^n) \pm o(x^m) = o(x^k)$ *where* $k = \min\{n, m\}$.
(5) $o(x^n)^m = o(x^{nm})$

Proof. 1. $\lim_{x\to 0} x^n/x^m = 0$ only if $n > m$.

2. If $f(x) = o(x^n)$ then $\lim_{x\to 0} f(x)/x^n = 0$. If $m \leq n$, then $x^{n-m} \to 0$ or 1 as $x \to 0$ so

$$\lim_{x\to 0} \frac{f(x)}{x^m} = \lim_{x\to 0} x^{n-m}\frac{f(x)}{x^n} = \lim_{x\to 0} x^{n-m} \lim_{x\to 0} \frac{f(x)}{x^n} = 0$$

and $f(x) = o(x^m)$.

3. This is Theorem 1.3.

4. By property 2, we can replace $o(x^m)$ with $o(x^n)$ and Theorem 1.1 then implies,

$$o(x^n) \pm o(x^m) = o(x^n) \pm o(x^n) = o(x^n).$$

5. This can be established by applying property 3 and induction on m. It is obviously true for $m = 1$. Suppose it is true for $m = k$ and consider the case $m = k + 1$:

$$o(x^n)^{k+1} = o(x^n)^k o(x^n) = o(x^{nk})o(x^n) = o(x^{nk}x^n) = o(x^{n(k+1)}).$$

By induction the formula is true for all positive integers m. □

Note the strict inequality in Corollary 2.1. In fact, $x^n \neq o(x^n)$ since $x^n/x^n \to 1 \neq 0$ as $x \to a$.

Simplifying o-Symbols

Let $P(x)$ be a polynomial expression involving various powers of x^n, $o(x^m)^k$, and products of these, $x^n o(x^m)^k$. Then $P(x)$ can be reduced to an expression of the form

$$P(x) = a_0 x^{n_0} + o(x^{m_0}).$$

Furthermore,

$$o(P(x)) = o(x^k) \text{ where } k = \min\{n_0, m_0\}.$$

To see why this is true, perform the following steps. First, find the term that has the smallest power of x (only), say $a_0 x^{n_0}$. The sum of all the other terms involving only x can replaced by $o(x^{n_0})$ since they all have exponents greater than n_0 (apply Corollary 2.1 and 2.4). Thus $P(x)$ can be replaced with an expression of the form $a_0 x^{n_0} + Q(x)$ where all the terms of $Q(x)$ involve some o-symbol.

Next simplify $Q(x)$ to $o(x^{m_0})$ by replacing any terms like $c \cdot o(x^m)^k$ with $o(x^{mk})$, replacing any terms like $c \cdot x^n o(x^m)^k$ with $o(x^{n+mk})$ and replacing any sums like $o(x^m) + o(x^n)$ with $o(x^k)$ where $k = \min\{m, n\}$ (apply Corollary 2.2–5).

The final conclusion above can be shown by considering which exponent, n_0 or m_0, is smaller. If $m_0 < n_0$ then $a_0 x^{n_0} = o(x^{m_0})$ so $P(x)$ reduces to $P(x) = o(x^{m_0})$ and $o(P(x)) = o(o(x^{m_0})) = o(x^{m_0})$.

If $m_0 \geq n_0$, then $o(P(x)) = o(a_0 x^{n_0} + o(x^{m_0}))$ stands for a function $f(x)$ satisfying

$$\lim_{x\to 0} \frac{f(x)}{a_0 x^{n_0} + h(x)} = 0$$

Here $h(x) = o(x^{m_0})$, so $h(x) = o(x^{n_0})$ and $h(x)/x^{n_0} \to 0$ as $x \to 0$. Therefore,

$$\lim_{x\to 0} \frac{f(x)}{x^{n_0}} = \lim_{x\to 0} \frac{f(x)}{a_0 x^{n_0} + h(x)} \Big(a_0 + \frac{h(x)}{x^{n_0}} \Big) = 0 \cdot (a_0 + 0) = 0.$$

Therefore, $f(x) = o(x^{n_0})$ and we conclude $o(P(x)) = o(x^{n_0})$ as claimed.

Example. Let $P(x) = (x - x^2 + o(x - x^2))^2$. We simplify the o-symbol first: $o(x - x^2) = o(x + o(x)) = o(x)$. Then,

$$P(x) = (x - x^2 + o(x))^2 = (x - x^2)^2 + 2(x - x^2)o(x) + o(x)^2$$

Let's deal with each term one at a time:

$$\begin{aligned} (x - x^2)^2 &= x^2 - 2x^3 + x^4 = x^2 + o(x^2) \\ 2(x - x^2)o(x) &= 2xo(x) - 2x^2 o(x) = o(x^2) + o(x^3) = o(x^2) \\ o(x)^2 &= o(x^2) \end{aligned}$$

Therefore,

$$P(x) = x^2 + o(x^2) + o(x^2) + o(x^2) = x^2 + o(x^2)$$

and

$$o(P(x)) = o(x^2 + o(x^2)) = o(x^2).$$

Example.

$$\begin{aligned} &\cos(\sin(x)) = 1 - \frac{1}{2}\sin^2(x) + \frac{1}{4!}\sin^4(x) + o(\sin^5(x)) \\ &= 1 - \frac{1}{2}\big(x - \frac{1}{3!}x^3 + o(x^4)\big)^2 + \frac{1}{4!}\big(x - \frac{1}{3!}x^3 + o(x^4)\big)^4 + o\Big(\big(x - \frac{1}{3!}x^3 + o(x^4)\big)^5\Big) \\ &= 1 - \frac{1}{2}\big(x^2 - \frac{1}{3}x^4 + o(x^5)\big) + \frac{1}{4!}\big(x^4 + o(x^5)\big) + o(x^5 + o(x^6)) \\ &= 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + o(x^5) \end{aligned}$$