Peter Cholak Math 221 Wednesday, December 18, 1996. Final

This final is worth 150 points. There are 7 problems on 5 pages and one blank page. *Be sure to show all your work; you will not get full credit unless your work matches your answer!!*

(10 points) Let $A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$.

If b is some real number, use Cramer's rule to find the x_3 coordinate of the solution to the following equation:

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ -1 \\ -3 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & -3 \end{bmatrix}$. (10 points) Determine whether $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ is consistent and if so, find the general solution.

(10 points) Determine whether $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}$ is consistent and if so, find the general solution.

(40 points – 8 points each) For each of the following, indicate whether the statement is true or false. Include a brief explanation of your answer. *You will not receive full credit without a reasonable explanation!* The rule $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 - x_2 y_2$ defines an inner product on the vector space $\mathbb{R}^2$. (Here $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$.) Let W be all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = 1$ and

V be all 2×2 matrices. Then W is a subspace of V. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & b \end{bmatrix}$.

Then for all values of b the linear transformation $T_A(\vec{x}) = A\vec{x}$ is onto. Let W

be the set of $\vec{b}$ such that $A\vec{x} = \vec{b}$ is consistent, where A is an 4×3 matrix. Then the W is a subspace of $\mathbb{R}^4$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If

$k_1\vec{x}_1 + k_2\vec{x}_2 = \vec{0}$ then $k_1 T(\vec{x}_1) + k_2 T(\vec{x}_2) = \vec{0}$.

(15 points) Consider the following three vectors in $\mathbb{R}^4$ (with the Euclidean inner product): $(1,0,0,1),(0,1,0,-1)$ and $(0,0,1,1)$. Find an *orthonormal* basis for the space spanned by these three vectors (you will need to use Gram-Schmidt).

Consider the following system of linear equations:

$$\begin{array}{ccccccc} 1x_1 & + & 4x_2 & + & 0x_3 & = & a_1 \\ 0x_1 & + & 2x_2 & + & 1x_3 & = & a_1 \\ 0x_1 & + & 0x_2 & + & 2x_3 & = & a_1 \end{array}$$

(5 points) Find the coefficient matrix P of this system of linear equations.

(10 points) Is P invertible? If so, find the inverse matrix P^{-1}. Otherwise, explain why P has no inverse.

(5 points) Find the *general* solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to this system of linear equations.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

(15 points) Find the eigenvalues for the matrix A. For each eigenvalue that you find, find a basis for the corresponding *eigenspace*.

(5 points) Find a matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Consider the vectors $\vec{v} = \dfrac{1}{\sqrt{2}}(1, -1, 0)$ and $\vec{w} = (0, 0, 1)$. These vectors form an *orthonormal* basis for a subspace W in $\mathbb{R}^3$ (with the Euclidean inner product). Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by orthogonally projecting onto the subspace W; so $P(\vec{x}) = \text{Proj}_W(\vec{x})$.

(10 points) We know that for a vector $\vec{x}$, $P(\vec{x}) = \text{Proj}_W(\vec{x}) = (\vec{x} \cdot \vec{w})\vec{w} + (\vec{x} \cdot \vec{v})\vec{v}$. Compute the standard matrix associated with P. (Hint: compute $P(\vec{e}_1)$, $P(\vec{e}_2)$ and $P(\vec{e}_3)$.)

(For the remaining parts of the problem there is nothing to compute.) (5 points) $\vec{z} = (1, 1, 0)$ is orthogonal to the subspace W. Is the vector $\vec{z}$ an eigenvector for P? If so, what is the eigenvalue?

(5 points)Is the vector $\vec{v}$ an eigenvector for P? If so, what is the eigenvalue. What about $\vec{w}$? 5 points) Find the eigenvalues of the standard matrix for P.

Find bases for the eigenspaces. (Hint: use the above two parts.)