Math 221 Section 1 **Fall semester 1997-98**

Test No. 3
Solutions

1.(20 points) Use Cramer's formulas to solve the following system of equations

$$\begin{aligned} x - y \quad &= 3 \\ y + z &= 3 \\ x - y + z &= 2 \end{aligned}$$

Warning: No credit will be given for any other method of solution.

Solution.

$$x = \frac{\begin{vmatrix} 3 & -1 & 0 \\ 3 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}} = 7 \quad y = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 3 & 3 & 2 \\ 1 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}} = 4 \quad z = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 3 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}} = -1.$$

2. (15 points). A sequence of matrices is defined as follows:

$$M_1 = [\,1\,],\ M_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ M_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \dots ,$$

$$M_n = \begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ 2n+1 & 2n+2 & \dots & 3n \\ \dots & \dots & \dots & \dots \\ (n-1)n+1 & (n-1)n+2 & \dots & (n-1)n+n \end{bmatrix}, \dots$$

Show that $\det M_n = 0$ for all $n \geq 3$. Why are the first two different?

Solution.

1. $n \geq 3$. The difference between the third and the second column is a column whose all entrees are 1, so is the difference between the second column and the first. By the second rule, the determinant equals 0, because it equals a determinant with two identical columns.

(It will also work with rows: the two identical rows will have all entrees equal n.)

The proof will not be valid for n =1, 2, because there will be no third row. Indeed, det(M_1)= 1, det(M_2)= -2.

3. (15 points) Use the big formula to evaluate the determinant $\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \end{vmatrix}$.

Write out explicitly all nonzero products with the proper sign.

Solution

The only nonzero products are $a_{11}\ a_{23}\ a_{32}\ a_{42} = 2$ and $a_{14}\ a_{23}\ a_{32}\ a_{41} = 6$. The first permutation is odd, the second is even, so the determinant equals -2+6=4.

4. (20 points) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$. What are the eigenvalues of A, A^2 and A^3 ?

Solution.

Because A is a triangular matrix the eigenvalues of A are the pivots: 1, -1, 3. The eigenvalues of A^2 are 1, 1, 9, the eigenvalues of A^3 are 1, -1, 27. If λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n with the same eigenvector.

Find an eigenvector corresponding to the eigenvalue 3 of A.

Solution.

We need to find a nonzero vector in the nullspace of the matrix $\begin{bmatrix} -2 & 2 & 2 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, e.g. (2,1,1)

[any multiple of this vector will do].

Is this an eigenvector of A^2 ? If so, to which eigenvalue of A^2 does it belong.

Solution.

Answer. Yes, it corresponds to the eigenvalue 9. Proof. Denoting, the eigenvector $\mathbf{x}$, we have $A\mathbf{x} = 3\mathbf{x}$, $A^2\mathbf{x} = A(3\mathbf{x}) = 3A\mathbf{x} = 9\mathbf{x}$.

Are these matrices diagonalizable? Justify.

Yes. The matrix A has three distinct eigenvalues and, therefore, three independent eigenvectors. Consequently it is diagonalizable. The other two have the same eigenvectors. Therefore, they are diagonalizable too. Note that A^2 has a double eigenvalue 1.

5. (15 points) The characteristic polynomial of the matrix $\begin{bmatrix} 8 & 3 & -3 \\ -6 & -1 & 3 \\ 12 & 6 & -4 \end{bmatrix}$ factors into $-(\lambda+1)(\lambda-2)^2$. Decide if the matrix is diagonalizable?

Solution.

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 2$. The eigenvalue -1 will supply one column of the matrix S. The eigenvalue 2 must supply 2 columns. So, the matrix is diagonalizable if there are 2 independent eigenvectors corresponding to the eigenvalue 2, and is not if there is only one independent eigenvector. The eigenvectors lie in the nullspace of the matrix

$\begin{bmatrix} 6 & 3 & -3 \\ -6 & -3 & 3 \\ 12 & 6 & -6 \end{bmatrix}$. This matrix is of rank 1 (row 2 = - row 1, row 3 = 2 row 1) so there are two independent eigenvectors, and the matrix is diagonalizable.

6. (20 points) The matrices $A = \begin{bmatrix} -1 & 0 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ have the same eigenvalues as the matrix in question 5. Which of these matrices is diagonalizable?

Solution

Again, the answer depends on number of independent eigenvectors with eigenvalue 2:

For A we look at the matrix $\begin{bmatrix} -3 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which is of rank 1. Diagonalizable.

For B we look at the matrix $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$, which is of rank 2. Not diagonalizable.

7. Extra credit. The eigenvalues of the matrix $A = \begin{bmatrix} 23 & -36 \\ -36 & 2 \end{bmatrix}$ are -25 and 50. Find the orthogonal matrix, which diagonalizes A..

Solution.

Finding the eigenvectors involves finding generators of the nullspaces of matrices

$\begin{bmatrix} 48 & -36 \\ -36 & 27 \end{bmatrix} \mathbf{x} = \mathbf{0}$ gives an eigenvector (4, 3) for the eigenvalue -25.

$\begin{bmatrix} -28 & -36 \\ -36 & -48 \end{bmatrix} \mathbf{x} = \mathbf{0}$ gives the other eigenvector (3, -4). The eigenvectors are orthogonal.

To get an orthogonal matrix S we must norm the vectors by dividing them by 5. So

$S = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{-4}{5} \end{bmatrix}$. Remember, This is not the only orthogonal matrix, which will do it.