Name: ______________________________
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Practice Exam 2

1. Give definitions for each of the following terms.

1. a *subspace* of a vector space
2. the *null space* of a matrix A
3. an *echelon matrix* U
4. the *free variables* in a matrix equation $U\vec{x} = \vec{0}$, where U is an echelon matrix.
5. *homogeneous* equation and *homogeneous* solution.
6. the *rank* of a matrix A
7. *linearly independent* vectors
8. the *span* of some vectors
9. a *basis* of a vector space
10. the *dimension* of a vector space
11. *orthogonal* subspaces of $\mathbb{R}^n$
12. the *orthogonal complement* of a subspace of $\mathbb{R}^n$

2. Decide which of the following sets of vectors are subspaces. Explain your reasoning. (Make *sure* that you indicate what conditions must be satisified for a set of vectors to be a subspace).

(a). Is $\{(a-b, a+b, 3a-2b) \ : \ a, b \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$?

(b). Is $\{(a, 2b, 3a+2b, b+1) \ : \ a, b \in \mathbb{R}\}$ a subspace of $\mathbb{R}^4$?

3. Construct a 2×4 matrix whose null space is the *span* of $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ in $\mathbb{R}^4$. What is the rank of your matrix?

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 2 & 0 & 5 & 0 & \frac{13}{2} \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix}.$$

(a). Find the *null space* of A.

(b). If possible give the general solution to $A\vec{x} = \vec{b}$ when $\vec{b} = (1, 1, 3, 2)$ and when $\vec{b} = (1, 1, -1, 0)$.

5.

(a). Find an equation of the form
$$ax + by + cz = 0$$
for the *plane* in $\mathbb{R}^3$ consisting of all linear combinations $\{t(1, \frac{1}{2}, \frac{1}{3}) + s(1, \frac{1}{4}, \frac{1}{5}) \ : \ t, s \in \mathbb{R}\}$. (**Hint:** Consider the augmented matrix
$$\begin{bmatrix} A & \vec{\beta} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \beta_1 \\ \frac{1}{2} & \frac{1}{4} & \beta_2 \\ \frac{1}{3} & \frac{1}{5} & \beta_3 \end{bmatrix}.$$
Now find conditions on the β_i so that $A\vec{x} = \vec{\beta}$ has a solution.)

(b). Find an equation of the form
$$ax + by + cz + dw = 0$$
for the *"plane"* in $\mathbb{R}^4$ spanned by the vectors
$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$
.

6. Consider the matrix
$$A = \begin{bmatrix} 1 & t & 1 \\ t & s & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(a). For what values of s and t does A have rank 2? rank 3?

(b). If s and t are chosen so that A has rank 2, find the nullspace of A and the left nullspace of A.

7. Decide which of the following sets of vectors are *linearly independent.* Demonstrate your reasoning (don't just write down "yes" or "no"!)

(a). $\{(1, 0, a, b), (0, 2, c, d), (0, 0, 1, e)\}$ where a, b, c, d, e are fixed constants.

(b). the *matrices* $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ regarded as *vectors.*

(c). the *columns* of the matrix $\begin{bmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ \frac{1}{2} & \frac{-1}{3} & 0 \end{bmatrix}$.

8. Find bases for the following spaces.

(a). The null space of the matrix $A = \begin{bmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

(b). Find a basis for the space of all upper triangular 3×3 matrices $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ *such that* $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in the nullspace of U.

9. Decide whether the following statements are True of False, and tell why.

(a). There is always a non-0 vector in the null space of the matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$.

(b). If the rank r of a matrix A is the same as the number of columns of A, then the equation $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^r$.

(c). If U is an echelon form of the matrix A, then $R(A) = R(U)$.

(d). If A is a 2×18 matrix, there are always at least 16 linearly independent vectors in the null space of A.

10. Find bases for the four subspaces associated with the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 2 & 2 \end{bmatrix}$.

11. Find a vector $\vec{v}$ in $\mathbb{R}^4$ which is orthogonal to the space spanned by

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}.$$

What is the *orthogonal complement* of the line through $\vec{v}$?