Name: ______________________________
November 19, 1997

Practice Exam 3

The exam will cover §4.2 to §6.2.

1. Give definitions for each of the following terms.

1. An orthogonal basis.
2. An orthonormal basis.
3. An orthogonal matrix.
4. The cofactor matrix of a matrix.
5. The cross product $\vec{v} \times \vec{w}$ of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$.
6. An eigenvector and its eigenvalue.

2. (Projections)

(a). Consider the vector $\vec{v} = (1, -2, -1, 2) \in \mathbb{R}^4$. Compute the projection matrix for $\vec{v}$ and compute the projection of $\vec{w} = (1, 0, 0, 0)$ and $\vec{u} = (a, a, b, b)$ onto the line through $\vec{v}$.

(b). The column space of $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}$ is a plane in $\mathbb{R}^3$. Find the edges of the projection of the square with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$. What kind of shape is the projection?

3. Find an orthonormal basis for the column space of the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. Write the vector $(4, 3, 4, 3)$ as a linear combination of your orthonormal basis.

4. Sketch brief arguments for the following statements:

(a). Any set $q_1, \ldots, q_m$ of orthonormal vectors is always linearly independent.

(b). Suppose that v_1, v_2, v_3 are eigenvectors for the matrix A with corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Assume that $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$, and $\lambda_2 \neq \lambda_3$. Show that the vectors v_1, v_2, v_3 are linearly independent.

5. Let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$. Compute Q^{-1}. Solve the equation $Q\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and solve the equation $Q\vec{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

6. Consider the matrix equation $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \alpha \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$. For what α is the z coordinate of the solution (x, y, z) equal to 0? (HINT: use Cramer's rule).

7. Let $A_1 = \begin{bmatrix} 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, $A_4 = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$, etc. So A_k is a $k \times k$ matrix, has 3's on the main diagonal, and has -1's immediately above and below the main diagonal.

(a). Compute $|A_1|$ and $|A_2|$.

(b). Using cofactors, obtain a formula for $|A_k|$ in terms of $|A_l|$ for $l < k$.

(c). Can you give an argument as to why $|A_k|$ is divisible by 3 when k is odd?

8. Decide whether the following are true or false; give a brief reason if true and an example if false.

(a). If the columns of the matrix A are orthogonal, then $A^T \cdot A$ is a *diagonal* matrix.

(b). The set of all eigenvectors with the same eigenvalue λ is a *subspace* of $\mathbb{R}^m$.

(c). If v_1 is an eigenvector with eigenvalue λ_1 and v_2 is an eigenvector with eigenvalue λ_2, then $v_1 + v_2$ is an eigenvector with eigenvalue $\lambda_1 + \lambda_2$.

9. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

(a). Find S so that $S^{-1}AS = \Lambda$ is a diagonal matrix.

(b). Compute A^{-5}. (Hint: compute Λ^{-5} and use S.)

10. Find the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1, 1)$, $(2, 4)$, $(1, 4)$, and $(2, 7)$.