Provide the precise definition of each of the following concept: The span of a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

All linear combinations of the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. All vectors of the form $c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$.

A set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ being linearly independent.

The only solution of to $c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$ is $c_i = 0$ for all i.

The elementary row operations.

Scaling (multiplying a row by a scalar), add a multiple of a row to another row, and switch two rows.

The determinant of an $n \times n$ matrix A.

The cofactor expansion along the first row. $\det(A) = \sum_{j=1}^{n} a_{1,j}(-1)^{j+1}\det(A_{1,j})$.

TRUE/FALSE. Determine whether the followings statements are true or false. Be sure to provide a reason for your answer. Let $T : \mathbf{R}^m \to \mathbf{R}^m$ be a linear transformation. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of linearly independent vectors in $\mathbf{R}^m$. Then the set of vectors $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is also linearly independent.

FALSE. Consider $T : \mathbf{R}^m \to \mathbf{R}^n$ by $T(\mathbf{x}) = \mathbf{0}$.

Let A and B be two invertible matrices. Then ABA^{-1} is invertible.

TRUE. $(AB^{-1}A^{-1})(ABA^{-1}) = I$.

If A and B are square and $AB = I$ then $\det(A) = 0$.

FALSE. A is invertible and hence $\det(A) \neq 0$.

Let $T(\mathbf{x})$ be a linear transformation. T is one to one if and only if the kernel of T is $\{0\}$.

TRUE. This was a lemma from class. $T(\mathbf{0}) = \mathbf{0}$. So if T is one to one then the kernel of T is $\{0\}$. Now assume the kernel of T is $\{0\}$ and T is not one to one. So $T(\mathbf{x}) = T(\mathbf{y})$ where $\mathbf{x} \neq \mathbf{y}$. $\mathbf{0} = T(\mathbf{x}) - T(\mathbf{y}) = T(\mathbf{x} - \mathbf{y})$. Hence $\mathbf{x} - \mathbf{y} \neq \mathbf{0}$ is in the kernel of T. Contradiction.

Let A be the following matrix:

$$A = abcdefghi$$

if $\det A = 1$ then (hint: use the properties of determinant) $\det abc$
$-2d - 2e - 2f$
$ghi = ?$

The above matrix is the result of scaling the second row of A by -2. This action multiples determinant of A by -2. So the determinant is -2.

$\det abc$
$a + db + ec + f$
$ghi = ?$

The above matrix is the result of adding the first row to the second row of A. This action does not affect the determinant. The determinant is 1.

Describe the null space of the matrix: $A = 1 - 11 - 1$
$- 22 - 13$
$3 - 231$
$4 - 44 - 4$. Find all solutions of the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = -2$
8
4
$- 8$.

To save time we will row-reduce the matrix $[A|\mathbf{b}]$. It reduces to 10024
010410
00114
00000. Hence a spanning set for the null space is -2
$- 4$
$- 1$
1. A solution for $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_p 4$
10
4
0 and all the solutions are in the form $\mathbf{x}_p + \mathbf{x}$, where $\mathbf{x}$ is in the null space of A.

Write 1
4 as a linear combination of 1
1 and -1
2.

We must solve the equation $x_1 1$
$1 + x_2 - 1$
$2 = 1$
4. The augmented matrix of this system is $1 - 11$
124 which row-reduces to 102
011. Hence 21
$1 + -1$
$2 = 1$
4.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation. If $T(1,1) = (2,3), T(-1,2) = (4,5)$ then $T(1,4) =$?.

$T(1,4) = T(2(1,1) + (-1,2)) = 2T(1,1) + T(-1,2) = 2(2,3) + (4,5) = (4,6) + (4,5) = (8,11)$, since T is a linear transformation.

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent vectors. Show that $\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent vectors.

Assume that $x_1\mathbf{v}_1 + x_2(\mathbf{v}_1 - \mathbf{v}_2) + x_3(\mathbf{v}_2 + \mathbf{v}_3) = \mathbf{0}$. Then $(x_1 + x_2)\mathbf{v}_1 + (-x_2 + x_3)\mathbf{v}_3 + x_3\mathbf{v}_3 = \mathbf{0}$. Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors, $x_1 + x_2 = 0$, $-x_2 + x_3 = 0$ and $x_3 = 0$. Hence the $x_0 = x_1 = x_2 = 0$ and therefore $\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent vectors