Exam 1 is *Wednesday, November 22* in class. It will primarily cover Chapter 4 (sections 4.1 though 4.7) and Chapter 5 (sections 5.1 though 5.4). Of cousre this material is based on the earlier material and hence you should be comfortable with the earlier material.

Another good source of practice problems for the exam can be found in the homework problems assigned (especially the uncollected ones, if you didn't do them yet ...). The quizes are also a good place to look.

The exam will have some definitions, some true/false with reasons problems, some computations problems and one or two short proof-like problems. Very similar to the last exam.

Prof. Cholak (office: 223 CCMB) will have drop in office hours during the day on Tuesday (on November 21 and 28) from 1 pm to 5 pm. Prof Wong (office: 234 CCMB) will have drop in office Tuesday (on November 21 and 28) from 8 pm to 10 pm.

*Possible definitions and explanatory material.*

1. If $\mathcal{B}$ is a basis for a vector space $V$, and $\vec{x}$ is a vector in $V$, what is the coordinate vector $[\vec{x}]_{\mathcal{B}}$? To which vector space does $[\vec{x}]_{\mathcal{B}}$ belong?

2. If $\mathcal{B}$ is a basis for $R^n$, what is meant by the *change-of-coordinates matrix* $P_{\mathcal{B}}$? What does $P_{\mathcal{B}}$ enable one to compute?

3. what is meant by a *finite dimensional* vector space? by an *infinite dimensional* vector space?

4. If $A$ is an $m \times n$ matrix, define the *row space* of $A$. Define the rank of $A$ (note that by Theorem 14 in §4.6 there are several equivalent definitions of the rank of $A$. That theorem is pretty important.)

5. You should know how to obtain "change-of-basis" matrices as in §4.7. If $\mathcal{B}$ and $\mathcal{C}$ are bases for a vector space $V$, let $P$ denote the change of basis matrix. For a vector $\vec{v}$ in $V$, you should be able to state the relationship between $[\vec{v}]_{\mathcal{B}}$ with $[\vec{v}]_{\mathcal{C}}$ using the matrix $P$.

6. If $A$ is a square $(m \times m)$ matrix, what is meant by an eigenvector for $A$? an eigenvalue? the characteristic polynomial?

7. You should be familiar with the following fact: if $\vec{v}_1, \ldots, \vec{v}_n$ are eigenvectors for $A$ with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_n$ (distinct means that if $i \neq j$ then $\lambda_i \neq \lambda_j$), then the vectors $\vec{v}_i$ are linearly independent.

8. If $A$ and $B$ are square matrices of the same size, what does it mean to say that $A$ and $B$ are *similar*? Be sure you know the difference between similarity of matrices (in this sense) and our older notion of (row) equivalent matrices.

9. If $A$ is a square matrix, what does it mean to say that $A$ is diagonalizable? There are several equivalent formulations. You should know how to obtain

the diagonal matrix similar to $A$ (eigenvalues...) as well as an invertible matrix $P$ which defines the necessary similarity transformation.

10. You should remember that if an $m \times m$ matrix has $m$ distinct eigenvalues it is diagonalizable. (why is that?) Moreover, you should have on hand examples of matrices which are *not* diagonalizble.

11. If $T : V \to W$ is a linear transformation and $\mathcal{B}$ and $\mathcal{C}$ are bases for $V$ and $W$ respectively, what is meant by the matrix of $T$ relative to $\mathcal{B}$ and $\mathcal{C}$? If $M$ is this matrix, you should be able to use $M$ to relate $[\vec{v}]_{\mathcal{B}}$ and $[T(\vec{v})]_{\mathcal{C}}$ for any vector $\vec{v}$ in $V$.

True/False: (as usual, if the statement is true give a reason, otherwise provide an example showing that it is false.)

1. Every set of 5 vectors in $R^4$ is linearly dependent.
2. There is a subspace $H$ of $R^5$ with $\dim H = 6$.
3. If $A$ is an $m \times m$ matrix where $m \geq 2$, and $A$ has only 1 eigenvalue, then $A$ is not diagonalizable.
4. If the $m \times m$ matrix $A$ is similar to $B$, then $I_m + A$ is similar to $I_m + B$.
5. If $v$ and $w$ are eigenvectors for the $m \times m$ matrix $A$ then $v + w$ is again an eigenvector for $A$.
6. If $v$ is an eigenvector for the $m \times m$ matrix $A$, then $v$ is an eigenvector for $A^m$ for any integer $m \geq 0$.
7. Let $A$ be a $5 \times 3$ matrix. If the rank of $A$ is 3, then $A$ defines a one-to-one linear transformation $R^3 \to R^5$.
8. If $A$ is a diagonalizable $4 \times 4$ matrix with eigenvalues $\lambda = 3$ and $\lambda = 0$, then $A$ has rank 2.

Let $A = 1012$
0130
0001. Find a basis for $Nul(A)$. What is the dimension of this subspace? Find a basis for $Col(A)$. What is the dimension of this subspace? What is the rank of $A$?

Let $M$ be a $8 \times 9$ matrix, and assume that the null space of $M$ is 4 dimensional. How many pivots does $M$ have? What is the dimension of the row space of $M$?

Let $H$ be the column space of $A$ where $= 1201$
0131
0010 Find a basis $\mathcal{B}$ for $H$. For each of the 4 columns $\vec{a_i}$ $(1 \leq i \leq 4)$ of $A$, find $[\vec{a_i}]_{\mathcal{B}}$.

Let $\mathcal{B} = \{11, 1 - 2\}$. Show that $\mathcal{B}$ is a basis for $R^2$ If $\vec{x}$ is a vector in $R^2$ with $[\vec{x}]_{\mathcal{B}} = 2$
1, find $\vec{x}$. If $\vec{x} = 2$
1, find $[\vec{x}]_{\mathcal{B}}$.

Let $\mathcal{E} = \{1, t, t^2, t^3\}$ be the standard basis for $\mathbf{P}_3$, the space of polynomials of degree $\leq 3$. Let $\mathbf{p}(t) = 1 + t^2 + t^3$, and $\mathbf{q}(t) = 1 + 2t + 3t^3$. Find the coordinate vectors $[\mathbf{p}(t)]_{\mathcal{E}}$ and $[\mathbf{q}(t)]_{\mathcal{E}}$. Explain how to use (a) to show that $\{\mathbf{p}(t), \mathbf{q}(t)\}$ form a linearly independent subset of $\mathbf{P}_3$.

Let $A, B, C$ be $m \times m$ matrices. Assume that $A$ and $B$ are similar, and that $B$ and $C$ are similar. Carefully prove that $A$ and $C$ are similar.

Let $\vec{u}_i$ be eigenvectors for $\lambda_i$, for $i = 0, 1, 2$. Assume that $\lambda_j \neq \lambda_k$. Show that $\vec{u}_0, \vec{u}_1, \vec{u}_2$ are linearly independent.

Let $A = 32$

22. Decide if $A$ is diagonalizable. If it is, find invertible $P$ and diagonal $D$ so that $A = PDP^{-1}$.

Consider the vector space $V = \mathbf{M}_2$ of $2\times2$ matrices. Let $B = \{E_{1,1} = 1000, E_{1,2} = 0100, E_{2,1} = 0010, E_{2,2} =$
Show that $\mathcal{B}$ is a basis for $V$. (So what is $\dim V$?) Let $A = 11$

02. Let $T : V \to V$ be given by the rule $T(M) = AM - M$. Show that $T$ is a linear transformation. Compute the matrix $N$ of $T$ relative to the basis $\mathcal{B}$. Note that $N$ is a $4 \times 4$ matrix, not a $2 \times 2$ matrix!!! Find a basis of the nullspace of $N$, and find the corresponding "vectors" in $V$ which lie in the kernel of $T$.

Let $0 \neq h$ be a constant, and consider the matrix A = 1 1 1/h 1
0 2 h 1
0 0 1 1
0 0 0 3
. For what values of $h$ does the $\lambda = 1$ eigenspace have dimension 2?

If the $\lambda = 1$ eigenspace has dimension 2, is $A$ diagonalizable? Why?

Find a basis for the $\lambda = 3$ eigenspace of $A$.

Consider the basis $\mathcal{B}$ of $R^2$ whose vectors are $\vec{b_1} = 1$
$2 + \sqrt{5}$ and $\vec{b_2} = 1$
$2 - \sqrt{5}$. Let A = -2 1
1 2 . Find the $\mathcal{B}$ coordinate vectors $[A\vec{b_1}]_{\mathcal{B}}$ and $[A\vec{b_2}]_{\mathcal{B}}$ . Find the matrix of $A$ relative to the basis $\mathcal{B}$. Let $\vec{v} = \vec{b_1} + 2\vec{b_2} = 3$
$6 - \sqrt{5}$. Solve the equation $A\vec{x} = \vec{v}$ for $\vec{x} = c_1\vec{b_1} + c_2\vec{b_2}$.

Let $A$ be a $5 \times 3$ matrix, and let $\vec{v_1} = 1$
2
0
0
0
, $\vec{v_2} = 0$
1
3
0
0
, $\vec{v_3} = 0$
0
1
4
0
. Suppose that the equations $A\vec{x} = \vec{v_i}$ have solutions for $i = 1, 2, 3$. Prove that the nullspace of $A$ is $\{0\}$. (Hint: what can be said about the rank of $A$?)