November 17, 2004

Solutions: Practise Exam 2

2.

1. False: every set of 5 or more vectors in $\mathbb{R}^4$ is linearly dependent since the dimension of $\mathbb{R}^4$ is 4.

2. False: a subspace has smaller ($\leq$) dimension.

3. False: the identity matrix is a counter example.

4. True: $P(I_m + A)P^{-1} = I_m + PAP^{-1} = I_m + B$.

5. True: the eigenvectors form a subspace.

6. True: if v is an eigenvector for the eigenvalues λ of A then it is an eigenvector for the eigenvalues λ^m of A^m

7. True: by the Rank Theorem the dimension of null space = n - rank = 3 - 3 = 0.

8. True: it is clear the rank of the matrix:

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is 2 (this problem is slightly too hard).

3. It is clear that the matrix is row equivalent to the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

thus x_3 is free and the null space is spanned by the vector $(-1, -3, 1, 0)$. The column space is spanned by the 3 vectors $(1, 0, 0), (0, 1, 0), (2, 0, 1)$. The rank of $A = 3$.

4. The rank of $A = 9 - 4 = 5$ so there are 5 pivots and the dimension of the row space is 5.

5. A basis is given by the vectors $\mathcal{B} = \{\vec{a}_1 = (1, 0, 0), \vec{a}_2 = (2, 1, 0), \vec{a}_3 = (0, 3, 1)\}$. It is clear that $[\vec{a}_1]_\mathcal{B} = (1, 0, 0), [\vec{a}_2]_\mathcal{B} = (0, 1, 0), [\vec{a}_3]_\mathcal{B} = (0, 0, 1)$. The vector $(1, 1, 0) = -\vec{a}_1 + \vec{a}_2$ so $[(1, 1, 0)]_\mathcal{B} = (-1, 1, 0)$.

6. (a) It is clear that

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$$

so the two column vectors are linearly independent. (b) $\vec{x} = 2(1, 1) + (1, -2) = (3, 0)$. (c) $(2, 1) = 5/3(1, 1) + 1/3(1, -2)$ thus $[(2, 1)]_\mathcal{B} = (5/3, 1/3)$.

7. (a) $[\mathbf{p}]_\mathcal{E} = [1, 1, 1]^T, [\mathbf{q}]_\mathcal{E} = [1, 2, 3]^T$ (b) the two vectors are linearly independent because $[1, 1, 1]^T, [1, 2, 3]^T$ are not multiple of each other.

8. $A = PBP^{-1}, B = QCQ^{-1}$ so $A = PQCQ^{-1}P^{-1} = (PQ)C(PQ)^{-1}$.

9. This is a Theorem in the textbook.

10. The characteristic polynomial is $(3-\lambda)(2-\lambda)-4=\lambda^2-5\lambda+2$. The two eigenvalues are $\lambda_1=(5+\sqrt{17})/2, \lambda_2=(5-\sqrt{17})/2$. An eigenvector of λ_1 is obtained by taking a solution of the matrix

$$\begin{bmatrix} \frac{1-\sqrt{17}}{2} & 2 \\ 2 & \frac{-1-\sqrt{17}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for example,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{17}}{4} \\ 1 \end{bmatrix}.$$

An eigenvector of λ_2 is obtained by taking a solution of the matrix

$$\begin{bmatrix} \frac{1+\sqrt{17}}{2} & 2 \\ 2 & \frac{-1+\sqrt{17}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for example,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1+\sqrt{17}}{4} \end{bmatrix}.$$

The matrix P is formed by taking these 2 eigenvectors as columns:

$$P = \begin{bmatrix} \frac{1+\sqrt{17}}{4} & 1 \\ 1 & -\frac{1+\sqrt{17}}{4} \end{bmatrix}.$$

(The numbers here are fairly bad and this problem would not OK for an exam.)

11. (a) The condition

$$c_1E_{11}+c_2E_{12}+c_3E_{21}+c_4E_{22}=\vec{0}$$

is equivalent to the condition

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e., $c_1=c_2=c_3=c_4=0$ so $\dim V=4$.

(b) $T(M_1+M_2)=A(M_1+M_2)-(M_1+M_2)=AM_1-M_1+AM_2-M_2=T(M_1)+T(M_2); T(cM)=A(cM)-cM=c(AM-M)=cT(M)$.

(c) The matrix of T is obtaind by consider the image of the basis under T:

$$T(E_{11})=\begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} = -E_{12}-2E_{22},$$

$$T(E_{12})=\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = E_{11}-2E_{22}$$

$$T(E_{21})=\begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix} = -E_{12}+2E_{21}-2E_{22}$$

$$T(E_{22})=\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = -E_{11}.$$

Thus the matrix of T is given by

$$\begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ -2 & -2 & -2 & 0 \end{bmatrix}.$$

(d) The matrix is invertible so the null space is trivial.

12. (a) and (b) Obviously we have to exclude $h = 0$. Next we find the null space of the matrix

$$A - I_4 = \begin{bmatrix} 0 & 1 & 1/h & 1 \\ 0 & 1 & h & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1/h & 0 \\ 0 & 0 & h - 1/h & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

thus we see that the dimension of the column space is 3 unless $h = 1$ and for $h = 1$ the dimension of the column space is 2. By Rank Theorem we conclude that the dimension of the null space is 2 if and only if $h = 1$ (and if $h \neq 1$ then the dimension of the null space is 1). If $h = 1$ then the matrix is diagonalizable because we then get 4 linearly independent eigenvectors, 2 for the eigenvalue 1 and one each of the eigenvalues 2 and 3. (c) for $\lambda = 3$ we consider the null space of the matrix

$$A - 3I_4 = \begin{bmatrix} -2 & 1 & 1/h & 1 \\ 0 & -1 & h & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 0 & 2 + (1/h + h)/2 \\ 0 & -1 & 0 & 1 + h/2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so x_4 is a free variable and so $[1 + (1/h + h)/4, 1 + h/2, 1/2, 1]^T$ is an eigenvector.

13. (a) $A\vec{b}_1 = [\sqrt{5}, 5 + 2\sqrt{5}]^T = \sqrt{5}\vec{b}_1$ so $[A\vec{b}_1]_{\mathcal{B}} = [\sqrt{5}, 0]^T$. $A\vec{b}_2 = [-\sqrt{5}, 5 - 2\sqrt{5}]^T = -\sqrt{5}\vec{b}_2$ so $[A\vec{b}_2]_{\mathcal{B}} = [0, -\sqrt{5}]^T$. (b) the matrix representing the linear transformation relative to the basis $\mathcal{B}$ is given by

$$\begin{bmatrix} \sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{bmatrix}$$

(c) this means we are solving the equation

$$\begin{bmatrix} \sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the solution is clearly $c_1 = 1/\sqrt{5}, c_2 = -2/\sqrt{5}$.

14. The 3 given vectors are linearly independent which shows that the column space is 3 dimension, i.e., the rank of the matrix is 3 hence the dimension of the null space is 0 because $\dim nul(A) + 3 = 3$.