(4 points each.) Provide the precise definition of each of the following concept: The *basis* of a vector space V.

B is a basis for V iff the span of B is V and S is linearly independent.

A *finite-dimensional* vector space V.

A vector space that has a finite basis.

The *eigenvalues* of a matrix A.

λ is an eigenvalue if there is a non-zero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$.

Two matrices A and B are *similar*.

A and B are similar iff there is a invertible matrix P such that $A = PBP^{-1}$.

TRUE/FALSE. Determine whether the followings statements are true or false. Be sure to provide a reason for your answer. (2 points each for correct answer; 2 points each for correct reason.) A is invertible iff $\lambda = 0$ is not a eigenvalue of A.

TRUE. λ is an eigenvalue of A iff the nullspace of $A - \lambda$ is $\mathbf{0}$ (the vector space with just the zero vector). So 0 is an eigenvalue of A iff nullspace of A is not $\mathbf{0}$ iff A is not invertible.

Let A be a $m \times n$ matrix. Then the column space of A plus the nullspace of A^T is n.

FALSE. The rank theorem tells us that the rank of A plus the dimension of the nullspace of A is n. So the rank of A^T plus the dimension of the nullspace of A is m. The rank of A^T, rank of A, dimension of the column space of A and the rowspace of A are the same.

The dimension of P^3 (all polynomials of degree 3 or less) is 3.

FALSE. $\{1, t, t^2, t^3\}$ is a basis for P^3 hence P^3 has dimension 4.

All diagonalizable matrices are invertible.

FALSE. $A10$
00 is diagonalizable not invertible.

(20 points.) Let $A = 1021$
0101
1122
. Row reduce A.

Find a basis for the column space of A. What the dimension of the column space A?

Find a basis for the row space of A. What the dimension of the row space A?

Find a basis for the null space of A. What the dimension of the null space A?

A row reduces to 1021
0101
0000
. The first two columns of A form a basis for the column space of A which has dimension 2. The first two rows of A form a basis for the row space of A which has dimension 2. $(-2, 0, 1, 0)$ and $(-1, -1, 0, 1)$ form a basis for the null space of A which also has dimension 2.

(20 points.) Let $A = 2d1$
022
003
.

For what values of d is A diagonalizable? For any such value, diagonalize A (do not find P^{-1}).

The goal is to diagonalize A. The eigenvalues are 2 and 3. This is only possible if eigenspace corresponding the $\lambda = 2$ has dimension 2. If $\lambda = 2$, $A - \lambda I = 0d1$
002
003
. So the eigenspace corresponding the $\lambda = 2$ has dimension 2 iff $d = 0$. Lets assume $d = 0$. Then a basis for eigenspace corresponding the $\lambda = 2$ is $(1, 0, 0), (0, 1, 0)$. If $\lambda = 1$ (and $d = 0$), $A - \lambda I = -101$
$0 - 12$
000
which row reduces to $10 - 1$
$01 - 2$
000
. So a basis for eigenspace corresponding the $\lambda = 3$ is $(1, 2, 1)$. Hence $P = 101$
012
001
and $D = 200$
020
003.

(15 points.) Consider the vector space $V = \mathbf{M}_2$ of 2×2 matrices. Let B $= \{E_{1,1} = 1000, E_{1,2} = 0100, E_{2,1} = 0010, E_{2,2} = 0001\}$.

From the sample exam (problem #12) we know $\mathcal{B}$ is a basis for V.

Let $T : V \to V$ be given by the rule $T(M) = M - M^T$. Show that T is a linear transformation. Compute the matrix N of T relative of the basis $\mathcal{B}$. (As in problem #12 N is a 4×4 matrix, not a 2×2 matrix.)

(a) $T(cM_1 + dM_2) = (cM_1 + dM_2) - (cM_1 + dM_2)^T = (cM_1 + dM_2) - c(M_1)^T + d(M_2)^T = c(M_1 - (M_1)^T) + d(M_2 - (M_2)^T) = cT(M_1) + dT(M_2)$. So T is a linear transformation.

(b) $T(E_{1,1}) = \vec{0}$, $T(E_{1,2}) = E_{1,2} - E_{2,1}$, $T(E_{2,1}) = E_{2,1} - E_{1,2}$ and $T(E_{2,2}) = \vec{0}$.

$$N = 000001100 - 1 - 100000.$$

(15 points.) Let A be a 5×8 matrix whose rank is 5. Show that $A\vec{x} = \vec{b}$ is consistent and has infinitely many solutions for all $\vec{b}$ in R^5.

The column space of A has dimension 5. Since the column space of A is a subspace of $\mathbf{R}^5$ and the dimension of $\mathbf{R}^5$ is 5, the column space of A is $\mathbf{R}^5$. Hence for every $\vec{b}$ in R^5 is consistent. The nullspace of A has dimension $8-5 = 3$. Thus the equation $A\vec{x} = \vec{0}$ has infinitely solutions. Therefore there are infinitely many solutions for all equations $A\vec{x} = \vec{b}$.